CAHIER 25-2001

MULTIPLE PUBLIC GOODS AND LEXICOGRAPHIC PREFERENCES : REPLACEMENT PRINCIPLE

Lars EHLERS

Centre de recherche
et développement en économique

C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7

Téléphone : (514) 343-6557
Télécopieur : (514) 343-5831
crde@crde.umontreal.ca
http://www.crde.umontreal.ca/

Université de Montréal

CAHIER 25-2001

MULTIPLE PUBLIC GOODS AND LEXICOGRAPHIC PREFERENCES : REPLACEMENT PRINCIPLE

Lars EHLERS[1]

[1] Centre de recherche et développement en économique (C.R.D.E.) and Département de sciences économiques, Université de Montréal

October 2001

The author acknowledges useful comments and suggestions by Hans Peters, Ton Storcken, William Thomson, and two anonymous referees. He thanks the seminar participants at CentER, Tilburg University, The Netherlands, April 1999, and the participants of the conference on Axiomatic Resource Allocation Theory, Namur, Belgium, July 1999 for their helpful remarks. Part of the paper was written during the author's visit at the Department of Economics, University of Rochester. Financial support of the Netherlands Organization for Scientific Research (NWO) is acknowledged.

RÉSUMÉ

Nous étudions ici le problème de positionnement de deux biens publics pour un groupe d’agents avec des préférences unimodales sur un intervalle. Une alternative spécifie un emplacement pour chaque bien public. Dans Miyagawa (1998), chaque agent consomme seulement son bien public préféré sans rivalité. Nous étendons les préférences de manière lexicographique et caractérisons les classes de préférences à sommet unique par l’optimalité au sens de Pareto et la domination par remplacement. Ce résultat est assez différent de la caractérisation correspondante faite par Miyagawa (2001a).

Mots clés : préférences unimodales, biens publics multiples, lexicographique, domination par remplacement

ABSTRACT

We study the problem of locating two public goods for a group of agents with single-peaked preferences over an interval. An alternative specifies a location for each public good. In Miyagawa (1998), each agent consumes only his most preferred public good without rivalry. We extend preferences lexicographically and characterize the class of single-peaked preference rules by *Pareto-optimality* and *replacement-domination*. This result is considerably different from the corresponding characterization by Miyagawa (2001a).

Key words : single-peaked preferences, multiple public goods, lexicographic, replacement-domination

1 Introduction

Hotelling (1929) considers two competing businesses choosing where to locate on a street. He assumes that the businesses are identical and each individual patronizes only the one that is closest to where he lives. Miyagawa (1998) is the first who studies this model from a normative prospective and identifies rules on the basis of desirable properties. He considers the problem of a state government having to choose two locations where to build two identical public facilities. An alternative specifies for each of the two public goods a location. Agents have single-peaked preferences on some interval of possible locations and consume the public goods without rivalry: given two alternatives, an agent prefers an alternative to another if there is a location which he prefers to each of the locations of the other alternative. We call this extension of single-peaked preferences from the set of possible locations to the set of alternatives its *max-extension*.[1]

There are environments in which agents compare alternatives differently. At each point of time when an individual desires to consume the public good, he uses exactly one public good and therefore he has a single-peaked preference relation over the interval. However, sometimes it is not possible for him to consume the public good at his most preferred location. This could be due to several reasons, for example the good is used by other agents and therefore congested, or the good at his most preferred location is out of service. But primarily each agent consumes the good at his most preferred location. One example is where the town government locates two identical libraries on a street. If a certain book is not available at the first choice library of an individual who wants to borrow it, then he has to consume his second choice library. In these contexts we propose the *lexicographic-extension* of preferences[2]: given two alternatives, first an agent compares the most preferred locations of each of the two alternatives, and if there is a tie, then he compares the other locations. It turns out that this feature of preferences brings about results that are considerably different

[1]The model is further studied in Ehlers (2001) and Miyagawa (2001a,b). Further studies of the location of multiple public goods with different preferences are Barberà and Beviá (1999) and Bogomolnaia and Nicoló (1999).

[2]Dutta and Massó (1997) study two-sided matching when workers have lexicographic preferences. Each worker first compares firms and second co-workers.

from Miyagawa (2001a).

A basic requirement is *Pareto-optimality*, meaning that only efficient alternatives are chosen. *Pareto-optimality* is stronger in Miyagawa (2001a) than in our model. Indeed, except for preference profiles at which all agents have the same peak, each alternative that is *Pareto-optimal* with respect to the max-extension is also *Pareto-optimal* with respect to the lexicographic-extension.

Our main property is a notion of fairness. If the environment of an economy changes, then the welfares of all agents who are not responsible for this change are affected in the same direction: either all weakly gain or all weakly lose. As a variable parameter of an economy which may change over time, we consider preferences. Solidarity applied to such situations says that when the preference relation of an agent changes, then the welfares of all other agents are affected in the same direction. This replacement principle is called *welfare-domination under preference-replacement*, or simply *replacement-domination*.[3]

In different settings the "replacement principle" has been studied.[4] It seems to be a general feature of this property that in any model any class of rules characterized by *replacement-domination* and certain other properties is restricted. The review of Thomson (1999) supports this statement. For two pure public goods and the max-extension, Miyagawa (2001a) shows that there are only two rules satisfying *Pareto-optimality* and *replacement-domination*: the left-peaks rule and the right-peaks rule (for more details see Section 3). When considering the lexicographic-extension of preferences and therefore weakening *Pareto-optimality*, we show that *Pareto-optimality* and *replacement-domination* admit a large class of rules.

Each rule satisfying these properties is described by means of a fixed continuous and single-peaked binary relation over the set of locations. For each preference profile such a rule chooses one location to be a most preferred peak in the peak profile according to the fixed single-peaked relation. The second location is indifferent to

[3]Moulin (1987) introduces *replacement-domination* in the context of binary choice with quasi-linear preferences. He calls it "agreement".

[4]It has been studied in private good economies with single-peaked preferences (Barberà, Jackson, and Neme, 1997; Thomson, 1997), in classical exchange economies (Sprumont and Zhou, 1999), in economies with indivisible goods and monetary transfers (Thomson, 1998), and in one public good economies (Thomson, 1993; Vohra, 1999; Ehlers and Klaus, 2001).

this peak according to the fixed single-peaked relation such that, if *Pareto-optimality* is not violated, the locations belong to opposite sides of the peak of the fixed relation. We call these rules single-peaked preference rules and characterize them by *Pareto-optimality* and *replacement-domination*.

The organization of the paper is as follows. Section 2 introduces the general model and the axioms. Section 3 presents the definition and the characterization of the single-peaked preference rules. Section 4 contains the proof.

2 The Model

Let $N \equiv \{1, \ldots, n\}$, $n \in \mathbb{N}$, be the set of agents. Each agent $i \in N$ is equipped with a single-peaked and continuous preference relation R_i over $[0,1]$. By I_i we denote the indifference relation associated with R_i, and by P_i the corresponding strict preference relation. *Single-peakedness* means that there exists a location, called the *peak* of R_i and denoted by $p(R_i)$, such that for all $x, y \in [0,1]$, if $x < y \leq p(R_i)$ or $x > y \geq p(R_i)$, then yP_ix. By $\mathcal{R}$ we denote the set of all single-peaked preferences over $[0,1]$, and by $\mathcal{R}^N$ the set of *(preference) profiles* $R \equiv (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $S \subseteq N$, R_S denotes the restriction $(R_i)_{i \in S}$ of $R \in \mathcal{R}^N$ to S. Given $R \in \mathcal{R}^N$, $\underline{p}(R)$ denotes the smallest peak in the profile $(p(R_i))_{i \in N}$, and $\overline{p}(R)$ the greatest peak in the profile $(p(R_i))_{i \in N}$.

We choose the locations for two identical public goods in $[0,1]$. Let $M \equiv \{1,2\}$. Each agent has the freedom to choose the public goods he prefers. Therefore, the order in which we locate the facilities is irrelevant. An *alternative* is a tuple $x \equiv (x_1, x_2)$ such that $0 \leq x_1 \leq x_2 \leq 1$. We denote by $[0,1]^M$ the set of alternatives. Note that $(1,0)$ is *not* an alternative.

Each agent compares two alternatives via the *lexicographic preference relation* over $[0,1]^M$ induced by his single-peaked preference relation over [0,1].

Lexicographic-Extension of Preferences: Let $i \in N$ and $R_i \in \mathcal{R}$. Given two alternatives $x, y \in [0,1]^M$ and two permutations τ, ρ of M such that $x_{\tau(1)} R_i x_{\tau(2)}$ and $y_{\rho(1)} R_i y_{\rho(2)}$, *$x$ is lexicographically strictly preferred to y* if and only if either $x_{\tau(1)} P_i y_{\rho(1)}$ or ($x_{\tau(1)} I_i y_{\rho(1)}$ and $x_{\tau(2)} P_i y_{\rho(2)}$). Furthermore, *$x$ is lexicographically indifferent to y* if and only if $x_{\tau(1)} I_i y_{\rho(1)}$ and $x_{\tau(2)} I_i y_{\rho(2)}$. ○

Abusing notation, we use the same symbols to denote preferences over possible locations and lexicographic preferences over alternatives. When we extend preferences lexicographically, weak upper contour sets are neither closed nor open, and non-convex. Furthermore, indifference sets only contain a finite number of alternatives. Figure 1 illustrates this fact.

[Figure 1 enters around here.]

We also introduce Miyagawa's *max-extension* of preferences from [0,1] to $[0,1]^M$.

Max-Extension of Preferences, R_i^{max}**:** Let $i \in N$ and $R_i \in \mathcal{R}$. Given two alternatives $x, y \in [0,1]^M$ and two permutations τ, ρ of M such that $x_{\tau(1)} R_i x_{\tau(2)}$ and $y_{\rho(1)} R_i y_{\rho(2)}$, *$x$ is maximally strictly preferred to y*, $x P_i^{max} y$, if and only if $x_{\tau(1)} P_i y_{\rho(1)}$. Furthermore, *$x$ is maximally indifferent to y*, $x I_i^{max} y$, if and only if $x_{\tau(1)} I_i y_{\rho(1)}$. ○

Remark 2.1 In Figure 1 the closure of the weak upper contour set of R_1 at (0.2,0.6) is the weak upper contour set of R_1^{max} at (0.2,0.6). It is easy to see that this true for all $x \in [0,1]^M$ such that $p(R_1) \notin \{x_1, x_2\}$. If $p(R_1) \in \{x_1, x_2\}$, then the indifference set of R_1^{max} at x consists of the two line segments $[0, 0.5] \times \{0.5\}$ and $\{0.5\} \times [0.5, 1]$. If $x_1 = 0.5$, then the weak upper contour set of the lexicographic R_1 at x consists of the two line segments $[1 - x_2, 0.5] \times \{0.5\}$ and $\{0.5\} \times [0.5, x_2]$. As we will show, this "slight" change in weak upper contour sets and indifference sets brings about conclusions that are considerably different from those in Miyagawa (2001a). ◁

A *(decision) rule* is a mapping φ that associates with each $R \in \mathcal{R}^N$ an alternative, denoted by $\varphi(R) = (\varphi_1(R), \varphi_2(R))$. *Pareto-optimality* says that for each preference profile the chosen alternative cannot be changed in such a way that no agent is worse off and some agent is better off. Given $S \subseteq N$ and $R \in \mathcal{R}^N$, let $E(R_S)$ denote the set of *Pareto-optimal* (or efficient) alternatives for R_S. Formally, $E(R_S) = \{y \in [0,1]^M \mid \text{for all } x \in [0,1]^M, \text{ if for some } i \in S,\ xP_i y, \text{ then for some } j \in S,\ yP_j x\}$.

Pareto-Optimality: For all $R \in \mathcal{R}^N$, $\varphi(R) \in E(R)$.

For *Pareto-optimality* to hold it is not sufficient that for each public good the selected location belongs to $[\underline{p}(R), \overline{p}(R)]$. For every chosen alternative it is necessary that the closed interval having as two endpoints the two selected locations contains at least one peak. The straightforward proof is left to the reader.

Lemma 2.2 *Let φ be a rule. Then φ satisfies Pareto-optimality if and only if for all $R \in \mathcal{R}^N$ the following holds: (i) $\varphi_1(R), \varphi_2(R) \in [\underline{p}(R), \overline{p}(R)]$, and (ii) there exists $i \in N$ such that $p(R_i) \in [\varphi_1(R), \varphi_2(R)]$.*

By Lemma 2.2, the set of efficient alternatives depends only on the peaks of the profile.

Remark 2.3 For all $R \in \mathcal{R}^N$, let $E(R^{max})$ denote the set of *Pareto-optimal* alternatives in $[0,1]^M$ when we extend preferences maximally. It is easy to see that for all $x \in [0,1]^M$, $x \in E(R^{max})$ if and only if (i) $x_1, x_2 \in [\underline{p}(R), \overline{p}(R)]$ and (ii) for some $i, j \in N$, $p(R_i), p(R_j) \in [x_1, x_2]$, $x_1 P_i x_2$, and $x_2 P_j x_1$. For the lexicographic-extension of preferences, *Pareto-optimality* is weaker than for the max-extension. For all $R \in \mathcal{R}^N$ such that $\underline{p}(R) < \overline{p}(R)$, $E(R^{max}) \subset E(R)$. Generally the set $E(R)$ is considerably larger than $E(R^{max})$. For example, let $R \in \mathcal{R}^N$ be such that $\{p(R_i) \mid i \in N\} = \{0, 1\}$. Then $E(R^{max}) = \{(0,1)\} \subset ([0,1] \times \{1\}) \cup (\{0\} \times [0,1]) = E(R)$. $\lhd$

The solidarity property we discuss is *welfare-domination under preference-replacement*, or for short *replacement-domination*, introduced by Moulin (1987). It requires that when the preference relation of some agent changes, the welfares of all other agents are affected in the same direction.

Replacement-Domination: For all $j \in N$, and all $R, \bar{R} \in \mathcal{R}^N$ such that $R_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$, either [for all $i \in N\backslash\{j\}$, $\varphi(R) R_i \varphi(\bar{R})$] or [for all $i \in N\backslash\{j\}$, $\varphi(\bar{R}) R_i \varphi(R)$].

3 Single-Peaked Preference Rules

Miyagawa (2001a) shows that when $n \geq 4$ and we extend preferences from [0,1] to alternatives maximally, only the following two rules satisfy *Pareto-optimality* and *replacement-domination*.[5]

Left-Peaks Rule, L: For all $R \in \mathcal{R}^N$, if $\underline{p}(R) = \overline{p}(R)$, then $L(R) \equiv (\underline{p}(R), \underline{p}(R))$, and otherwise, $L(R) \equiv (\underline{p}(R), \min\{p(R_j) \mid j \in N \text{ and } \underline{p}(R) < p(R_j)\})$.

Right-Peaks Rule, G: For all $R \in \mathcal{R}^N$, if $\underline{p}(R) = \overline{p}(R)$, then $G(R) \equiv (\overline{p}(R), \overline{p}(R))$, and otherwise, $G(R) \equiv (\max\{p(R_j) \mid j \in N \text{ and } p(R_j) < \overline{p}(R)\}, \overline{p}(R))$.

By Lemma 2.2, the left-peaks rule and the right-peaks rule satisfy *Pareto-optimality*. However, both rules violate *replacement-domination* when agents compare alternatives lexicographically.

Example 3.1 Let $n \geq 3$ and $R \in \mathcal{R}^N$ be such that $p(R_1) = 0$, $p(R_2) = \frac{1}{2}$, $p(R_3) = 1$, and for all $i \in N \backslash \{1, 2, 3\}$, $p(R_i) \in \{0, 1\}$. Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_{N \backslash \{2\}} = R_{N \backslash \{2\}}$ and $p(\bar{R}_2) = \frac{2}{3}$. Then $L(R) = (0, \frac{1}{2})$ and $L(\bar{R}) = (0, \frac{2}{3})$. In particular, $L(R) P_1 L(\bar{R})$ and $L(\bar{R}) P_3 L(R)$. Thus, the left-peaks rule violates *replacement-domination*. Similarly, the right-peaks rule violates *replacement-domination*. $\lhd$

A "constant" rule selecting for each preference profile the same alternative satisfies *replacement-domination*, but not *Pareto-optimality*. Therefore, in our model *Pareto-optimality* and *replacement-domination* are independent.

Each rule satisfying *Pareto-optimality* and *replacement-domination* is described by a continuous and single-peaked binary relation over $[0, 1]$. Here is an example of such a rule when the single-peaked preference relation is continuous and symmetric around the peak $\frac{1}{3}$.

Example 3.2 We represent the symmetric single-peaked preference relation with peak $\frac{1}{3}$ by its corresponding indifference map. Let $f : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$ be such that for all $x \in [0, \frac{2}{3}]$, $f(x) \equiv \frac{2}{3} - x$. Then $f(\frac{1}{3}) = \frac{1}{3}$. For all $R \in \mathcal{R}^N$, we define the rule ϕ^f as follows: (a) if $\underline{p}(R) > \frac{1}{3}$, then $\phi^f(R) \equiv (\underline{p}(R), \underline{p}(R))$; (b) if $\overline{p}(R) < \frac{1}{3}$, then $\phi^f(R) \equiv (\overline{p}(R), \overline{p}(R))$; and (c) if $\frac{1}{3} \in [\underline{p}(R), \overline{p}(R)]$ and $j \in \{t \in N \mid$ for all $i \in N$,

[5]In the trivial case, when all peaks coincide, a rule satisfying *Pareto-optimality* and *replacement-domination* only needs to locate one good at the unanimous peak. We refer to Miyagawa (2001a) for the details.

$|p(R_i)-\frac{1}{3}| \geq |p(R_t)-\frac{1}{3}|\}$, then $\phi^f(R) \in \{(p(R_j), f(p(R_j))), (f(p(R_j)), p(R_j))\}$. When $p(R_1) \leq \ldots \leq p(R_l) \leq \frac{1}{3} \leq p(R_{l+1}) \leq \ldots \leq p(R_n)$, Figure 2 illustrates Case (c) for $j = l$. $\triangleleft$

[Figure 2 enters around here.]

Before we formally define our rules, we introduce an equivalent representation of a single-peaked preference relation over $[0, 1]$.

Let $R_0 \in \mathcal{R}$. Then $0R_01$ or $1P_00$. Suppose that $0R_01$. Since R_0 is continuous, for some $b \in [p(R_0), 1]$, $0I_0b$. For all $x \in [0, b]$, let $f(x) \in [0, b]$ be such that $xI_0f(x)$ and the following holds: (i) when $x \leq p(R_0)$, $f(x) \geq p(R_0)$, and (ii) when $x \geq p(R_0)$, $f(x) \leq p(R_0)$. Because R_0 is continuous, it follows that f is continuous. Therefore, with R_0 we associate a unique function $f : [0, b] \to [0, b]$ such that f is continuous, $f = f^{-1}$ (this follows from R_0 being a preference relation), and f is strictly decreasing (this follows from single-peakedness of R_0). In particular, f possesses as a unique fixed point $p(R_0)$, i.e. $f(p(R_0)) = p(R_0)$. Furthermore, associated with such a function is a unique single-peaked preference relation on [0,1].

Let $f : [0, b] \to [0, b]$ (or, alternatively, $f : [b, 1] \to [b, 1]$) be a continuous strictly decreasing function such that $f(0) = b$ ($f(b) = 1$) and $f = f^{-1}$ (f is symmetric). Denote by a its unique fixed point and by $\mathcal{F}$ the set of all such functions.

Single-Peaked Preference Rules, ϕ^f: Given $f \in \mathcal{F}$, the *single-peaked preference rule* ϕ^f based on f is defined as follows. For all $R \in \mathcal{R}^N$ such that $p(R_{i_1}) \leq \cdots \leq p(R_{i_n})$,

- if $a \notin [\underline{p}(R), \overline{p}(R)]$, then

$$\phi^f(R) \equiv \begin{cases} (\underline{p}(R), \underline{p}(R)) & \text{when } a < \underline{p}(R), \\ (\overline{p}(R), \overline{p}(R)) & \text{when } \overline{p}(R) < a. \end{cases}$$

- if $p(R_{i_l}) \leq a \leq p(R_{i_{l+1}})$, then

$$\phi^f(R) \equiv \begin{cases} (p(R_{i_l}), f(p(R_{i_l}))) & \text{when } f(p(R_{i_l})) \leq p(R_{i_{l+1}}), \\ (f(p(R_{i_{l+1}})), p(R_{i_{l+1}})) & \text{otherwise.} \end{cases}$$

Theorem 3.3 below says that if N contains at least 3 agents, then every decision rule satisfying *Pareto-optimality* and *replacement-domination* is a single-peaked preference rule. Section 4 contains the proof of Theorem 3.3. Note that when N contains only two agents, *replacement-domination* has no bite.

Theorem 3.3 *Let $n \geq 3$. Then the single-peaked preference rules are the only rules satisfying Pareto-optimality and replacement-domination.*

Remark 3.4 In Miyagawa (2001a) we have to distinguish two cases. If N contains at least four agents, then the left-peaks rule and the right-peaks rule are the only rules satisfying *Pareto-optimality* and *replacement-domination*. If N contains three agents, then any rule choosing for each profile two distinct peaks satisfies *Pareto-optimality* and *replacement-domination*. In Theorem 3.3, there is no distinction between these two cases. $\lhd$

Each single-peaked preference rule satisfies *anonymity* (the rule is symmetric in its arguments) and *coalitional strategy-proofness* (no group of agents can gain by jointly mispresenting their true preferences), as the careful reader may check. Therefore, other rules than rules choosing for each public good the corresponding location according to some median operation may satisfy *strategy-proofness* and additional axioms.[6] Note that we do not require the above properties, they are implied by *Pareto-optimality* and *replacement-domination*.

Finally we discuss the location of three public facilities. The result of Miyagawa (2001a) generalizes to these cases as follows:[7] If $n \geq 5$, then a rule satisfies *Pareto-optimality* and *replacement-domination* with respect to the max-extension if and only if either for all profiles the three different smallest peaks are chosen, or for all profiles the three different greatest peaks are chosen.

It is not obvious how to extend a single-peaked preference rule to the location of three goods. There are two single-peaked preference rules which can be extended in a straightforward way: it is the rule choosing for all profiles and all facilities the smallest peak (call this rule the *smallest-peak rule*) and the rule choosing for all profiles and all

[6]The first who characterized median solutions for one public good economies was Moulin (1980).

[7]Personal communication with E. Miyagawa at the *Fourth International Meeting of the Society for Social Choice and Welfare*, 1998, Vancouver, BC, Can.

facilities the greatest peak (call this rule the *greatest-peak rule*).[8] The smallest-peak rule and the greatest-peak rule satisfy *Pareto-optimality* and *replacement-domination* with respect to the lexicographic extension when we locate three facilities.

4 Proof of Theorem 3.3

Throughout this section let $n \geq 3$ and φ be a rule satisfying *Pareto-optimality* and *replacement-domination*. The following implications will be useful.

First, we prove that for any two efficient alternatives, if all agents are indifferent between them, then the two alternatives are the same.

Lemma 4.1 *For all $S \subseteq N$, all $R \in \mathcal{R}^N$, and all $x, y \in E(R_S)$, if for all $i \in S$, xI_iy, then $x = y$.*

Proof. Let $j \in S$ be such that $p(R_j) = \min_{i \in S} p(R_i)$. Since $x, y \in E(R_S)$, $p(R_j) \leq x_1 \leq x_2$ and $p(R_j) \leq y_1 \leq y_2$. Because xI_jy, it follows that $x_1 = y_1$ and $x_2 = y_2$. Hence, $x = y$. □

Second, if the preference relation of some agent changes and the choices of the rule at the initial and at the new profile are *Pareto-optimal* for the profile consisting of the remaining agents' preferences, then the same alternative is chosen for both profiles.

Lemma 4.2 *Let $j \in N$ and $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{N\setminus\{j\}} = \bar{R}_{N\setminus\{j\}}$. If $\varphi(R), \varphi(\bar{R}) \in E(R_{N\setminus\{j\}})$, then $\varphi(R) = \varphi(\bar{R})$.*

Proof. By *replacement-domination*, either [for all $i \in N\setminus\{j\}$, $\varphi(R)R_i\varphi(\bar{R})$] or [for all $i \in N\setminus\{j\}$, $\varphi(\bar{R})R_i\varphi(R)$]. Since $\varphi(R), \varphi(\bar{R}) \in E(R_{N\setminus\{j\}})$, then for all $i \in N\setminus\{j\}$, $\varphi(R)I_i\varphi(\bar{R})$. Hence, by Lemma 4.1, $\varphi(R) = \varphi(\bar{R})$. □

[8] For two public goods, the smallest-peak rule and the greatest-peak rule, respectively, are the single-peaked preference rules where the peak of the single-peaked preference relation is at 0 and at 1, respectively.

Third, if the preference relation of some agent changes and all *Pareto-optimal* alternatives at the new profile are also efficient for the profile consisting of the remaining agents' preferences, then all these agents weakly prefer the alternative chosen by the rule for the new profile to the initially chosen alternative.

Lemma 4.3 *Let $j \in N$ and $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$. If $E(\bar{R}) = E(R_{N\backslash\{j\}})$, then for all $i \in N\backslash\{j\}$, $\varphi(\bar{R}) R_i \varphi(R)$.*

Proof. By *replacement-domination*, either [for all $i \in N\backslash\{j\}$, $\varphi(\bar{R}) R_i \varphi(R)$] or [for all $i \in N\backslash\{j\}$, $\varphi(R) R_i \varphi(\bar{R})$]. Suppose that the assertion of Lemma 4.3 does not hold. Thus, for all $i \in N\backslash\{j\}$, $\varphi(R) R_i \varphi(\bar{R})$, and for some $h \in N\backslash\{j\}$, $\varphi(R) P_h \varphi(\bar{R})$. Because $E(\bar{R}) = E(R_{N\backslash\{j\}})$, $\varphi(\bar{R}) \in E(R_{N\backslash\{j\}})$. The previous two facts constitute a contradiction. □

Successive applications of Lemma 4.2 yield the following lemma.

Lemma 4.4 *Let $R \in \mathcal{R}^N$ be such that $|\{p(R_i) \,|\, i \in N\}| \leq n-1$. For all $\bar{R} \in \mathcal{R}^N$, if $\{p(\bar{R}_i) \,|\, i \in N\} = \{p(R_i) \,|\, i \in N\}$, then $\varphi(\bar{R}) = \varphi(R)$.*

The next lemma is an important step of the proof of Theorem 3.3. It says that for any preference profile, the open interval having as endpoints the two chosen locations contains no peak.

Lemma 4.5 *For all $R \in \mathcal{R}^N$ and all $i \in N$, $p(R_i) \notin]\varphi_1(R), \varphi_2(R)[$.*

Proof. Suppose that for some $R \in \mathcal{R}^N$ and some $j \in N$,

$$p(R_j) \in]\varphi_1(R), \varphi_2(R)[. \tag{1}$$

Without loss of generality, we suppose that $j \notin \{1,2\}$, $p(R_1) = \underline{p}(R)$, and $p(R_2) = \overline{p}(R)$. By successive applications of Lemma 4.2 we may assume that for all $i \in N\backslash\{1,2,j\}$, $p(R_i) = p(R_j)$.

Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_{N\backslash\{2\}} = R_{N\backslash\{2\}}$ and $\bar{R}_2 = R_j$. Thus, $E(\bar{R}) = E(R_{N\backslash\{2\}})$. By Lemma 4.3, for all $i \in N\backslash\{2\}$,

$$\varphi(\bar{R}) R_i \varphi(R). \tag{2}$$

Claim 1: $\varphi_2(\bar{R}) = \overline{p}(\bar{R}) = p(R_j)$.

Proof of Claim 1. Suppose that

$$\varphi_2(\bar{R}) < p(R_j). \tag{3}$$

Let $R'_j \in \mathcal{R}$ be such that $\overline{p}(R)P'_j\varphi_2(\bar{R})$ and $p(R'_j) = p(R_j)$, and let $\bar{R}' = (\bar{R}_{N\backslash\{j\}}, R'_j)$. By Lemma 4.4,

$$\varphi(\bar{R}') = \varphi(\bar{R}). \tag{4}$$

Let $R' = (R_{N\backslash\{j\}}, R'_j)$. If $p(R_j) \in \{\varphi_1(R'), \varphi_2(R')\}$, then by Lemma 4.3, $\varphi_2(\bar{R}') = p(R_j)$, which contradicts (3) and (4). Hence, by (1), *Pareto-optimality* and *replacement-domination*,[9]

$$\varphi_1(R') < p(R_j) < \varphi_2(R').$$

Thus, since $\overline{p}(R)P'_j\varphi_2(\bar{R})$ and by using (4),we obtain $\varphi(R')P'_j\varphi(\bar{R}')$, which contradicts Lemma 4.3. ◇

Let $\tilde{R} \in \mathcal{R}^N$ be such that $\tilde{R}_{N\backslash\{1\}} = R_{N\backslash\{1\}}$ and $\tilde{R}_1 = R_j$. By the same arguments as in Claim 1 it follows that $\varphi_1(\tilde{R}) = \underline{p}(\tilde{R}) = p(R_j)$. To summarize, Claim 1, the previous fact, *replacement-domination*, and (1) imply that

$$\varphi_1(\tilde{R}) = p(R_j),\ p(R_j) < \varphi_2(\tilde{R}),$$

and

$$\varphi_1(\bar{R}) < p(R_j),\ \varphi_2(\bar{R}) = p(R_j).$$

Let $\bar{R}^1 \in \mathcal{R}^N$ be such that $p(\bar{R}^1_1) = \underline{p}(\bar{R})$, for all $i \in N\backslash\{1\}$, $p(\bar{R}^1_i) = p(R_j)$, $\varphi_1(\bar{R})\bar{P}^1_2\varphi_2(\tilde{R})$, and $\varphi_2(\tilde{R})\bar{P}^1_3\varphi_1(\bar{R})$. Also, let $\bar{R}^2 \in \mathcal{R}^N$ be such that $\bar{R}^2_{N\backslash\{1\}} = \bar{R}^1_{N\backslash\{1\}}$ and $p(\bar{R}^2_1) = \overline{p}(\tilde{R})$. By Lemma 4.4,

$$\varphi(\bar{R}^1) = \varphi(\bar{R}) \text{ and } \varphi(\bar{R}^2) = \varphi(\tilde{R}). \tag{5}$$

By definition of $\bar{R}^1_2$ and $\bar{R}^1_3$, $\varphi(\bar{R})\bar{P}^1_2\varphi(\tilde{R})$ and $\varphi(\tilde{R})\bar{P}^1_3\varphi(\bar{R})$. Since $\bar{R}^1_{N\backslash\{1\}} = \bar{R}^2_{N\backslash\{1\}}$ and (5), the previous relations contradict *replacement-domination*. □

[9]Note that when $N = \{1, 2, 3\}$ we cannot conclude $\varphi(R') = \varphi(R)$.

The remaining proof of Theorem 3.3 is divided into two parts. In the first part we show Theorem 3.3 when N contains three agents. In the second part we use the three agents case to establish Theorem 3.3 for the general case.

Three Agents Case: $N = \{1, 2, 3\}$**.**

We show that φ satisfies *anonymity* and *peaks-onliness*.

Lemma 4.6 *φ satisfies* anonymity, *i.e. for all permutations σ of N, $\varphi(R) = \varphi(\sigma(R))$.*[10]

Proof. Let $R \in \mathcal{R}^N$ and $\sigma : N \to N$ be a permutation. If $|\{p(R_i) \mid i \in N\}| \in \{1, 2\}$, then the conclusion follows from Lemma 4.4. Suppose that $p(R_1) < p(R_2) < p(R_3)$. By Lemma 4.5, either $\varphi_1(R), \varphi_2(R) \in [p(R_1), p(R_2)]$ or $\varphi_1(R), \varphi_2(R) \in [p(R_2), p(R_3)]$. Without loss of generality, we suppose that

$$\varphi_1(R), \varphi_2(R) \in [p(R_1), p(R_2)].$$

Let $\bar{R} = (R_1, R_2, R_2)$. By Lemma 4.2, $\varphi(\bar{R}) = \varphi(R)$. By Lemma 4.4, $\varphi(\sigma(\bar{R})) = \varphi(\bar{R})$. Thus, by the two previous facts,

$$\varphi(\sigma(\bar{R})) = \varphi(R). \tag{6}$$

Next, we determine $\varphi(\sigma(R))$. If $\varphi_1(\sigma(R)), \varphi_2(\sigma(R)) \in [p(R_1), p(R_2)]$, then $\varphi(\sigma(R)) \in E(R_1, R_2)$. Hence, by Lemma 4.2, $\varphi(\sigma(R)) = \varphi(\sigma(\bar{R}))$. By (6), $\varphi(\sigma(R)) = \varphi(R)$, which is the desired conclusion.

Suppose that $\varphi_2(\sigma(R)) > p(R_2)$. Thus, by Lemma 4.5, $\varphi_1(\sigma(R)) \geq p(R_2)$. Let $\tilde{R} = (R_2, R_2, R_3)$. Thus, by Lemmas 4.2 and 4.4, $\varphi(\tilde{R}) = \varphi(\sigma(R))$. We distinguish three subcases.

Subcase 2.1: $\varphi_1(R) = \varphi_2(R) = p(R_2)$.

By Lemma 4.2, $\varphi(\tilde{R}) = \varphi(R)$. By Lemma 4.4, $\varphi(\tilde{R}) = \varphi(\sigma(R))$. The previous two equalities contradict $\varphi(R) \neq \varphi(\sigma(R))$.

Subcase 2.2: $\varphi_2(R) < p(R_2) < \varphi_1(\sigma(R))$.

Then $\varphi(\bar{R}) = \varphi(R)$ and $\varphi(\tilde{R}) = \varphi(\sigma(R))$. Using the same arguments as in the proof of Lemma 4.5 we derive a contradiction to *replacement-domination*.

Subcase 2.3: $\varphi_2(R) = p(R_2)$ (or $p(R_2) = \varphi_1(\sigma(R))$).

[10] As usual, $\sigma(R)$ is the permuted profile R according to σ.

By Subcase 2.1, $\varphi_1(R) < p(R_2)$. By Lemma 4.3, $\varphi(\tilde{R}) R_2 \varphi(R)$. Thus, $\varphi_1(\tilde{R}) = p(R_2)$. By Lemma 4.4, $\varphi(\tilde{R}) = \varphi(\sigma(R))$. Thus, $p(R_2) < \varphi_2(\tilde{R})$. Hence, by $\varphi(\bar{R}) = \varphi(R)$, $\varphi_1(\bar{R}) < \varphi_2(\bar{R}) = p(R_2) = \varphi_1(\tilde{R}) < \varphi_2(\tilde{R})$. Using the same arguments as in the proof of Lemma 4.5 we derive a contradiction to *replacement-domination*. □

Using similar arguments as in Lemma 4.6 it follows that φ satisfies *peaks-onliness*.

Lemma 4.7 *φ satisfies peaks-onliness, i.e. for all $R, \bar{R} \in \mathcal{R}^N$, if for all $i \in N$, $p(R_i) = p(\bar{R}_i)$, then $\varphi(R) = \varphi(\bar{R})$.*

We construct a function $f \in \mathcal{F}$ and show that $\varphi = \phi^f$. Before we define f we introduce additional notation. Given $x \in [0,1]$, $R^x \in \mathcal{R}^N$ denotes a preference profile such that $p(R^x_1) = 0$, $p(R^x_2) = x$, and $p(R^x_3) = 1$. *Pareto-optimality* implies that $\varphi_1(R^0) = 0$ or $\varphi_2(R^0) = 1$. Without loss of generality, we suppose that $\varphi_1(R^0) = 0$. The case $\varphi_2(R^0) = 1$ is symmetric by interchanging the roles of $\varphi_1(R^0)$ and $\varphi_2(R^0)$. Define $b \equiv \varphi_2(R^0)$ and the function $f : [0,b] \to [0,b]$ as follows.[11]

Definition of f: For all $x \in [0,b]$, when $x < \varphi_2(R^x)$, $f(x) \equiv \varphi_2(R^x)$, and when $x \geq \varphi_2(R^x)$, $f(x) \equiv \varphi_1(R^x)$.

We prove in three subsequent lemmas that $f \in \mathcal{F}$.

Lemma 4.8 *For all $x \in [0,b]$, $f(x) \in [0,b]$.*

Proof. Let $x \in [0,b]$. By Lemma 4.5, for all $i \in N$, $p(R^x_i) \notin]\varphi_1(R^x), \varphi_2(R^x)[$. Thus, if $\varphi_2(R^x) \in]b,1]$, then $\varphi_1(R^x) \in [x,1]$. Hence, $\varphi(R^0) P^0_1 \varphi(R^x)$ and $\varphi(R^x) P^0_3 \varphi(R^0)$, a contradiction to *replacement-domination*. Thus, $f(x) \in [0,b]$. □

Lemma 4.9 *If $x < x'$, then $f(x) > f(x')$. Moreover, $f = f^{-1}$.*

Proof. By Lemma 4.5, Lemma 4.8, and *replacement-domination*, for all $x \in]0,b[$, $f(x) \in]0,b[$. Thus, by Lemma 4.5, *Pareto-optimality*, and the definition of b, for all $x \in [0,b]$,

$$\varphi_1(R^x) = x \text{ or } \varphi_2(R^x) = x. \tag{7}$$

[11] When $\varphi_1(R^0) > 0$, we define $b \equiv \varphi_1(R^0)$ and a function $f : [b,1] \to [b,1]$.

Let $x, x' \in [0, b]$ be such that $x < x'$. Without loss of generality, we suppose that $\varphi_1(R^x) = x$. Thus, $\varphi_2(R^x) \geq x$. If $x' = \varphi_2(R^x)$, then Lemma 4.5, *replacement-domination* and (7) imply $\varphi(R^x) = \varphi(R^{x'})$. Hence, by definition of f, $f(x') = x < x' = f(x)$. If $x' \in]x, \varphi_2(R^x)[$, then by (7), $x' \in \{\varphi_1(R^{x'}), \varphi_2(R^{x'})\}$. Thus, by *replacement-domination*, $\varphi_1(R^{x'}), \varphi_2(R^{x'}) \in]x, \varphi_2(R^x)[$ and $f(x') < \varphi_2(R^x) = f(x)$. If $x' \in]\varphi_2(R^x), b]$, then by (7), $x' \in \{\varphi_1(R^{x'}), \varphi_2(R^{x'})\}$. Thus, by *replacement-domination*, $\varphi_1(R^{x'}) < x$ and $x' = \varphi_2(R^{x'})$. By definition of f, $f(x') = \varphi_1(R^{x'}) < x \leq f(x)$. Hence, f is strictly decreasing.

Since f is strictly decreasing, then f^{-1} is well-defined. For the second part, let $x \in [0, b]$. By (7) and the definition of f, $(x, f(x)) = \varphi(R^x)$ or $(f(x), x) = \varphi(R^x)$. By (7), $f(x) \in \{\varphi_1(R^{f(x)}), \varphi_2(R^{f(x)})\}$. Hence, by *replacement-domination*, $\varphi(R^{f(x)}) = \varphi(R^x)$. Thus, $f(x) = f^{-1}(x)$, the desired conclusion. □

Lemma 4.10 *The function f is continuous.*

Proof. It suffices to prove that f is left-continuous and right-continuous. We only show that f is left-continuous. Right-continuity can be similarly shown. Let $x \in [0, b]$ and $(x_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence converging to x. By Lemma 4.9, $(f(x_k))_{k \in \mathbb{N}}$ is a strictly decreasing sequence. Let $\bar{x} \equiv \lim_{k \to \infty} f(x_k)$. If $(f(x_k))_{k \in \mathbb{N}}$ does not converge to $f(x)$, then, by Lemma 4.9, $f(x) < \bar{x}$. We distinguish two cases.

Case 1: $x \leq f(x)$.

Consider $R^{\frac{1}{2}(f(x)+\bar{x})}$. By (7) in the proof of Lemma 4.9,

$$\frac{1}{2}(f(x) + \bar{x}) \in \{\varphi_1(R^{\frac{1}{2}(f(x)+\bar{x})}), \varphi_2(R^{\frac{1}{2}(f(x)+\bar{x})})\}.$$

Thus, by *replacement-domination*, $\varphi_1(R^{\frac{1}{2}(f(x)+\bar{x})}) < x$. Hence, for some $k \in \mathbb{N}$, $x_k \in]\varphi_1(R^{\frac{1}{2}(f(x)+\bar{x})}), x[$. Thus, $f(x_k) > \bar{x} > \varphi_2(R^{\frac{1}{2}(f(x)+\bar{x})})$. Because $\varphi(R^{x_k}) = (x_k, f(x_k))$ it follows that $\varphi(R^{\frac{1}{2}(f(x)+\bar{x}))}) P_1^{x_k} \varphi(R^{x_k})$ and $\varphi(R^{x_k}) P_3^{x_k} \varphi(R^{\frac{1}{2}(f(x)+\bar{x}))})$, which contradicts *replacement-domination*.

Case 2: $x > f(x)$.

Let $\varepsilon > 0$ be such that $f(x) + \varepsilon < \min\{\bar{x}, x\}$. By (7),

$$f(x) + \varepsilon \in \{\varphi_1(R^{f(x)+\varepsilon}), \varphi_2(R^{f(x)+\varepsilon})\}.$$

Thus, by *replacement-domination*, $\varphi_2(R^{f(x)+\varepsilon}) < x$. Hence, for some $k \in \mathbb{N}$, $x_k \in]\varphi_2(R^{f(x)+\varepsilon}), x[$. By our choice of ε, $f(x_k) > f(x) + \varepsilon$. Because $\varphi(R^{x_k}) = (x_k, f(x_k))$ it follows that $\varphi(R^{f(x)+\varepsilon}) P_1^{x_k} \varphi(R^{x_k})$ and $\varphi(R^{x_k}) P_3^{x_k} \varphi(R^{f(x)+\varepsilon})$, which contradicts *replacement-domination*. □

By Lemmas 4.8, 4.9, and 4.10, $f \in \mathcal{F}$. Let $a \in [0, b]$ be such that $f(a) = a$. The following lemma completes the proof of Theorem 3.3 for the three agents case.

Lemma 4.11 $\varphi = \phi^f$.

Proof. Let $R \in \mathcal{R}^N$. By *anonymity*, we may suppose that $p(R_1) \leq p(R_2) \leq p(R_3)$. We distinguish three cases.

Case 1: $a \in [\underline{p}(R), \overline{p}(R)]$.

Without loss of generality, we suppose that for some $j \in N$, $\phi^f(R) = (p(R_j), f(p(R_j)))$. Thus, $j \in \{1, 2\}$. If $j = 1$, then

$$\varphi(R^{p(R_1)}) = \varphi(R_3, R_{\{2,3\}}^{p(R_1)}) = \varphi(R_3, R_2^{p(R_1)}, R_2) = \varphi(R),$$

where the first equality follows from *replacement-domination* and $f(p(R_1)) \leq p(R_3)$, the second from *replacement-domination* and $p(R_2) \notin]p(R_1), f(p(R_1))[$, and the third from *anonymity* and *peaks-onliness*. Thus, by definition of f and ϕ^f, $\phi^f(R) = \phi^f(R^{p(R_1)}) = \varphi(R^{p(R_1)}) = \varphi(R)$, the desired conclusion.

Let $j = 2$. Then by *replacement-domination*, $f(p(R_2)) \leq p(R_3)$, *anonymity*, and *peaks-onliness*,

$$\varphi(R^{p(R_2)}) = \varphi(R_3, R_{\{2,3\}}^{p(R_2)}) = \varphi(R_3^{p(R_2)}, R_2, R_3). \tag{8}$$

By Lemma 4.9, $f = f^{-1}$ and $\varphi(R^{p(R_2)}) = \varphi(R^{f(p(R_2))})$. Thus, by using the same arguments as above and $p(R_1) \leq p(R_2) = \varphi_1(R^{f(p(R_2))})$,

$$\varphi(R^{p(R_2)}) = \varphi(R^{f(p(R_2))}) = \varphi(R_{\{1,2\}}^{f(p(R_2))}, R_1) = \varphi(R_2, R_2^{f(p(R_2))}, R_1) = \varphi(R_1, R_2, R_2^{f(p(R_2))}). \tag{9}$$

Now by (8), (9), and *replacement-domination*, $\varphi(R^{p(R_2)}) = \varphi(R)$. Thus, by definition of f and ϕ^f, $\phi^f(R) = \phi^f(R^{p(R_2)}) = \varphi(R^{p(R_2)}) = \varphi(R)$, the desired conclusion.

Case 2: $\overline{p}(R) < a$.

Let $\bar{R} \in \mathcal{R}^N$ be such that $p(\bar{R}_2) = 1$ and $\bar{R}_{N\backslash\{2\}} = R_{N\backslash\{2\}}$. By Case 1, $\varphi(\bar{R}) = (\overline{p}(R), f(\overline{p}(R)))$. Let $\bar{R}' \in \mathcal{R}^N$ be such that $p(\bar{R}'_2) = p(\bar{R}_1)$ and $\bar{R}'_{N\backslash\{2\}} = \bar{R}_{N\backslash\{2\}}$. By Lemma 4.3, for all $i \in N\backslash\{2\}$, $\varphi(\bar{R}')\bar{R}_i\varphi(\bar{R})$. Hence, by *peaks-onliness*, $\varphi(\bar{R}') = (\overline{p}(\bar{R}), \overline{p}(\bar{R}))$. Thus, by *replacement-domination*,

$$\varphi(R) = \varphi(\bar{R}') = (\overline{p}(R), \overline{p}(R)) = \phi^f(R),$$

the desired conclusion.

Case 3: $a < \underline{p}(R)$.

If $\underline{p}(R) \in \,]a, b]$, then the same proof of Case 2 yields the desired conclusion. Let $\underline{p}(R) \in \,]b, 1]$. Let $\bar{R} \in \mathcal{R}^N$ be such that $p(\bar{R}_2) = b$ and $\bar{R}_{N\backslash\{2\}} = R_{N\backslash\{2\}}$. By the previous fact, $\varphi(\bar{R}) = (b, b)$. Since $b < \underline{p}(R)$, *Pareto-optimality* implies $\varphi(R)\bar{P}_3\varphi(\bar{R})$. Thus, by *replacement-domination* and *peaks-onliness*, $\varphi_1(R) = \underline{p}(R)$. Suppose that $\varphi_2(R) > \underline{p}(R)$. Let $R' \in \mathcal{R}^N$ be such that $p(R'_2) = \frac{1}{2}(\underline{p}(R) + \varphi_2(R))$ and $R'_{N\backslash\{2\}} = R_{N\backslash\{2\}}$. By the previous argument, $\varphi_1(R') = \underline{p}(R)$. Hence, by Lemma 4.5, $\varphi_2(R') \leq p(R'_2) < \varphi_2(R)$. Thus, $\varphi(R')P_1\varphi(R)$ and $\varphi(R)P_3\varphi(R')$, a contradiction to *replacement-domination*. Therefore,

$$\varphi(R) = (\underline{p}(R), \underline{p}(R)) = \phi^f(R),$$

the desired conclusion. □

General Case: $N = \{1, \ldots, n\}$ **and** $n \geq 4$**.**

Let $\tilde{N} \equiv \{1, 2, 3\}$. We associate with φ a rule $\tilde{\varphi} : \mathcal{R}^{\tilde{N}} \to [0, 1]^M$ defined for three agents in the following way. For all $\tilde{R} \in \mathcal{R}^{\tilde{N}}$, let

$$\tilde{\varphi}(\tilde{R}) \equiv \varphi(\tilde{R}_1, \tilde{R}_2, (\tilde{R}_3)_{i \in N\backslash\{1,2\}}).$$

Obviously $\tilde{\varphi}$ inherits *Pareto-optimality* from φ.

Lemma 4.12 *$\tilde{\varphi}$ satisfies replacement-domination.*

Proof. Let $j \in \tilde{N}$ and $R, R' \in \mathcal{R}^{\tilde{N}}$ be such that $R_{\tilde{N}\backslash\{j\}} = R'_{\tilde{N}\backslash\{j\}}$. If $j \in \{1, 2\}$, then the assertion follows from *replacement-domination* of φ. Let $j = 3$. By definition of $\tilde{\varphi}$, $n \geq 4$, and Lemma 4.4, it follows that

$$\tilde{\varphi}(R) = \varphi(R_1, R_2, (R_3)_{i \in N\backslash\{1,2\}}) = \varphi(R_1, (R_2)_{i \in N\backslash\{1,3\}}, R_3) \quad (10)$$

and (by $R_{\tilde{N}\setminus\{3\}} = R'_{\tilde{N}\setminus\{3\}}$)

$$\tilde{\varphi}(R') = \varphi(R_1, R_2, (R'_3)_{i\in N\setminus\{1,2\}}) = \varphi(R_1, (R_2)_{i\in N\setminus\{1,3\}}, R'_3). \tag{11}$$

Hence, by (10) and (11), $\tilde{\varphi}$ inherits *replacement-domination* from φ. □

Because $\tilde{\varphi}$ satisfies *Pareto-optimality* and *replacement-domination*, the three agents case implies that there exists $f \in \mathcal{F}$ such that $\tilde{\varphi} = \tilde{\phi}^f$ (where $\tilde{\phi}^f : \mathcal{R}^{\tilde{N}} \to [0,1]^M$).

Let $R \in \mathcal{R}^N$. Then by *Pareto-optimality* and Lemma 4.5 there exists $j \in N$ such that $p(R_j) \in \{\varphi_1(R), \varphi_2(R)\}$. Let $k, h \in N$ be such that $p(R_k) = \underline{p}(R)$ and $p(R_h) = \overline{p}(R)$. Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_{\{j,k,h\}} = R_{\{j,k,h\}}$ and for all $i \in N\setminus\{j,k,h\}$, $\bar{R}_i = R_j$. Successive application of Lemma 4.2 yields $\varphi(\bar{R}) = \varphi(R)$. Let $\tilde{R} \in \mathcal{R}^N$ be such that $\tilde{R}_1 = \bar{R}_j$, $\tilde{R}_2 = \bar{R}_k$, and for all $i \in N\setminus\{1,2\}$, $\tilde{R}_i = \bar{R}_h$. Because $n \geq 4$, $|\{p(\bar{R}_i) \,|\, i \in N\}| \leq 3$, and $\{p(\bar{R}_i) \,|\, i \in N\} = \{p(\tilde{R}_i) \,|\, i \in N\}$, Lemma 4.4 yields $\varphi(\tilde{R}) = \varphi(\bar{R})$. Hence, $\varphi(R) = \varphi(\tilde{R})$.

Now by definition, $\varphi(\tilde{R}) = \tilde{\varphi}(\tilde{R}_{\{1,2,3\}})$. Thus, by $\tilde{\varphi} = \tilde{\phi}^f$ and $\varphi(R) = \varphi(\tilde{R})$, $\varphi(R) = \tilde{\phi}^f(\tilde{R}_{\{1,2,3\}})$. Because $\phi^f : \mathcal{R}^N \to [0,1]^M$ satisfies *anonymity* and for all $i \in N\setminus\{j,k,h\}$, $p(R_i) \in [\underline{p}(R), \varphi_1(R)] \cup [\varphi_2(R), \overline{p}(R)]$, it follows that $\phi^f(R) = \tilde{\phi}^f(\tilde{R}_{\{1,2,3\}})$. Hence, $\varphi(R) = \phi^f(R)$ and $\varphi = \phi^f$, the desired conclusion.

References

Barberà, S., and C. Beviá (1999): "Self-Selection Consistent Choices," Working Paper, Universitat Autònoma de Barcelona.

Barberà, S., M. Jackson, and A. Neme (1997): "Strategy-Proof Allotment Rules," *Games and Economic Behaviour 18*, 1–21.

Bogomolnaia, A., and A. Nicoló (1999): "Multiple Locations with Congestion: Normative Solutions," In: Logic, Game Theory and Social Choice, Proceedings of the International Conference, LGS '99 (ed. H. de Swart).

Dutta, B., and J. Massó (1997): "Stability of Matchings when Individuals have Preferences over Colleagues," *Journal of Economic Theory 75*, 464–475.

Ehlers, L. (2001): "Independence Axioms for the Provision of Multiple Public Goods as Options," *Mathematical Social Sciences 41*, 239–250.

Ehlers, L., and B. Klaus (2001): "Solidarity and Probabilistic Target Rules," *Journal of Public Economic Theory 3*, 167–184.

Hotelling, H. (1929): "Stability in Competition," *Economic Journal 39*, 41–57.

Miyagawa, E. (1998): "Mechanisms for Providing a Menu of Public Goods," Ph.D. Dissertation, University of Rochester.

Miyagawa, E. (2001a): "Locating Libraries on a Street," *Social Choice and Welfare 18*, 527–541.

Miyagawa, E. (2001b): "Mechanisms for Providing a Menu of Public Goods," Working Paper, Columbia University.

Moulin, H. (1980): "On Strategy-Proofness and Single-Peakedness," *Public Choice 35*, 437–455.

Moulin, H. (1987): "The Pure Compensation Problem: Egalitarian versus Laissez-Fairism," *Quarterly Journal of Economics 102*, 769–783.

Sprumont, Y., and L. Zhou (1999): "Pazner-Schmeidler Rules in Large Societies," *Journal of Mathematical Economics 31*, 321–339.

Thomson, W. (1993): "The Replacement Principle in Public Good Economies with Single-Peaked Preferences," *Economics Letters 42*, 31–36.

Thomson, W. (1997): "The Replacement Principle in Economies with Single-Peaked Preferences," *Journal of Economic Theory 76*, 145–168.

Thomson, W. (1998): "The Replacement Principle in Economies with Indivisible Goods," *Social Choice and Welfare 15*, 57–66.

Thomson, W. (1999): "Welfare-Domination under Preference-Replacement: a Survey and Open Questions," *Social Choice and Welfare 16*, 373–394.

Vohra, R.V. (1999): "The Replacement Principle and Tree Structured Preferences," *Economics Letters 63*, 175–180.

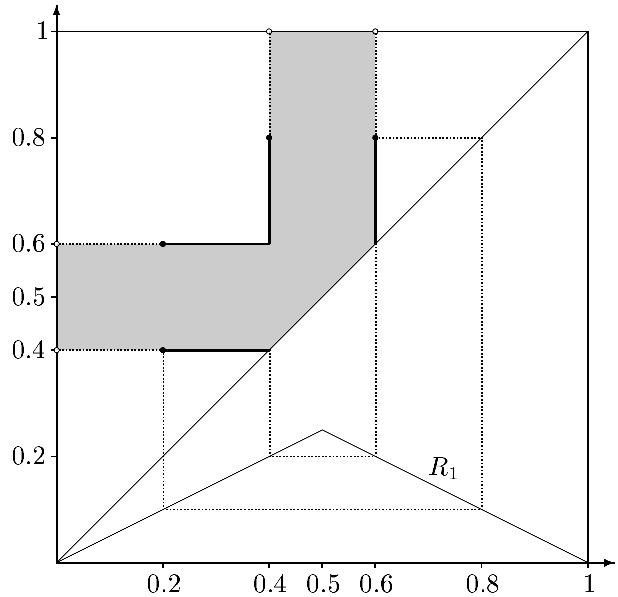

Figure 1: The preference R_1 is symmetric around the peak 0.5. The shaded area is the weak upper contour set at (0.2,0.6) when we extend R_1 lexicographically. Note that the dotted line segments $[0, 0.2[\times\{0.4\}$, $[0, 0.2[\times\{0.6\}$, $\{0.4\}\times]0.8, 1]$, and $\{0.6\}\times]0.8, 1]$ do not belong to this upper contour set. Furthermore, the four bullet points (0.2,0.4), (0.2,0.6), (0.4,0.8), and (0.6,0.8) are all alternatives in $[0, 1]^M$ that are indifferent to (0.2,0.6).

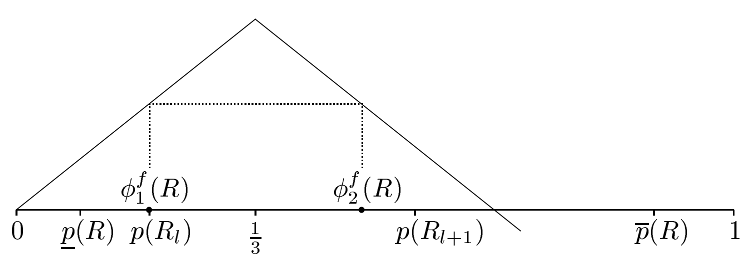

Figure 2: Illustration of Case (c) when $j = l$ in Example 3.2.