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ON FIXED-PATH RATIONING METHODS

Lars EHLERS

Centre de recherche
et développement en économique

C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7

Téléphone : (514) 343-6557
Télécopieur : (514) 343-5831
crde@crde.umontreal.ca
http://www.crde.umontreal.ca/

Université de Montréal

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Lars EHLERS[1]

[1] Centre de recherche et développement en économique (C.R.D.E.) and Département de sciences économiques, Université de Montréal

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RÉSUMÉ

Moulin (1999) caractérise les méthodes de rationnement suivant un sentier fixe par l'efficacité, la non-manipulation, l'homogénéité et la monotonicité de ressources. Nous donnons ici une preuve simple de son résultat.

Mots clés : méthodes de rationnement suivant un sentier fixe, préférences unimodales

ABSTRACT

Moulin (1999) characterizes the fixed-path rationing methods by *efficiency, strategy-proofness, consistency,* and *resource-monotonicity*. In this note, we give a straightforward proof of his result.

Key words : fixed-path rationing methods, single-peaked preferences

1 Introduction

We consider the problem of allocating a commodity among a group of agents with single-peaked preferences. For example, the commodity is a project requiring a certain number of hours of labor. One hour of labor is either infinitely divisible (the continuous model) or indivisible (the discrete model).

Sprumont (1991) is the paper that originated a number of axiomatic studies in the continuous model over the past ten years. Recently, Moulin (1999) introduces a new class of rules, called fixed-path rationing methods, and characterizes them by *efficiency*, *strategy-proofness*, *consistency*, and *resource-monotonicity*.[1] His theorem applies to both the continuous and the discrete model. The purpose of this note is to give a straightforward proof of his result.

2 The Model and the Result

Our formulation allows variations in the population and in the collective endowment. There is a finite set $\mathcal{N} = \{1, \ldots, n_0\}$ of potential agents.[2] Let Z denote the set of all (possible) endowments. The set Z is either $\mathbb{R}_+$ or $\mathbb{N} \cup \{0\}$. When $Z = \mathbb{R}_+$, we speak of the *continuous* model, and when $Z = \mathbb{N} \cup \{0\}$, we speak of the *discrete* model. For each agent $i \in \mathcal{N}$ there is an *a priori* fixed maximal consumption, denoted by $X_i \in Z \backslash \{0\}$. Given $X_i \in Z \backslash \{0\}$, agent i's consumption set is $[0, X_i] \cap Z$. From now on, given $b \in Z$, we write $[0, b]$ instead of $[0, b] \cap Z$. Thus, $[0, X_i]$ denotes agent i's consumption set. The vector $(X_i)_{i \in \mathcal{N}}$ of maximal consumption is fixed throughout. Given $N \subseteq \mathcal{N}$, let $X_N \equiv \sum_{i \in N} X_i$. Each agent $i \in \mathcal{N}$ is equipped with a preference relation R_i over $[0, X_i]$. Let P_i denote the strict preference relation associated with R_i. The preference relation R_i is *single-peaked* if there is a number $p(R_i) \in [0, X_i]$, called the *peak* of R_i, such that for all $x_i, y_i \in [0, X_i]$, if $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$, then $x_i P_i y_i$. Let $\mathcal{R}^i$ denote the set of all single-peaked preferences over $[0, X_i]$.

A collective endowment of a commodity has to be allocated among a finite set of agents. We allow the set of agents to be any (finite) subset $N \subseteq \mathcal{N}$. Let $\Omega \in Z$ denote

[1] In the continuous model, the first studies of *consistency* and *resource-monotonicity*, respectively, are Thomson (1994a,b).

[2] All results remain valid when $\mathcal{N}$ is a countable, infinite set.

the *endowment*. Given $N \subseteq \mathcal{N}$, a *(preference) profile* R is a list $(R_i)_{i\in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}^i$. Let $\mathcal{R}^N$ denote the set of all profiles for N. Given $R \in \mathcal{R}^N$, let $p(R) \equiv (p(R_i))_{i\in N}$. Given $S \subseteq N \subseteq \mathcal{N}$ and $R \in \mathcal{R}^N$, let $R_S \equiv (R_i)_{i\in S}$ denote the restriction of R to S. Given $N \subseteq \mathcal{N}$, an *economy* is a pair $(R,\Omega) \in \mathcal{R}^N \times [0, X_N]$. Let $\mathcal{E}^N \equiv \mathcal{R}^N \times [0, X_N]$. The economy $(R,\Omega) \in \mathcal{E}^N$ is in *excess demand* if $\Omega \leq \sum_{i\in N} p(R_i)$, and it is in *excess supply* if $\Omega > \sum_{i\in N} p(R_i)$.

For all $N \subseteq \mathcal{N}$, an *allocation* for $(R,\Omega) \in \mathcal{E}^N$ is a vector $z \in Z^N$ such that for all $i \in N$, $z_i \in [0, X_i]$, and $\sum_{i\in N} z_i = \Omega$, i.e., we do not allow free disposal. Let $Z(R,\Omega)$ denote the set of all allocations for (R,Ω). An allocation rule, or simply a *rule*, associates with each economy an allocation. Formally, a rule φ is a mapping $\varphi : \cup_{N\subseteq\mathcal{N}}\mathcal{E}^N \to \cup_{N\subseteq\mathcal{N}} Z^N$, such that for all $N \subseteq \mathcal{N}$ and all $(R,\Omega) \in \mathcal{E}^N$, $\varphi(R,\Omega) \in Z(R,\Omega)$.

A fixed-path rationing method (Moulin, 1999) relies on two fixed monotonic paths in the box $\times_{i\in\mathcal{N}}[0, X_i]$. For economies in excess demand, individual consumptions are computed along the first path, except that an agent whose demand is below his path-consumption receives exactly his demand. We apply a similar procedure for economies in excess supply by using the other path. We refer to Moulin (1999) for additional discussion.

Formally, given $N \subseteq \mathcal{N}$, an N-*path* is a mapping $g(N) : [0, X_N] \to Z^N$ such that[3]

(a) for all $\lambda \in [0, X_N]$, $\sum_{i\in N} g_i(N,\lambda) = \lambda$, and for all $i \in N$, $g_i(N,\lambda) \leq X_i$; and

(b) for all $\lambda, \tilde{\lambda} \in [0, X_N]$ such that $\lambda \leq \tilde{\lambda}$, for all $i \in N$, $g_i(N,\lambda) \leq g_i(N,\tilde{\lambda})$.

In the above definition, (a) is feasibility of an N-path and (b) is monotonicity of an N-path. For an N-path $g(N)$, let $\gamma(g(N))$ denote the range of $g(N)$, i.e., $\gamma(g(N)) \equiv \{g(N,\lambda) \mid \lambda \in [0, X_N]\}$.

A *full path* g specifies for each set $N \subseteq \mathcal{N}$ an N-path $g(N)$ (i.e. $g \equiv (g(N))_{N\subseteq\mathcal{N}}$) such that[4]

(c) for all $N \subseteq \tilde{N} \subseteq \mathcal{N}$, $\mathrm{proj}_N[\gamma(g(\tilde{N}))] = \gamma(g(N))$.

Condition (c) says that the projection of the range of the $\tilde{N}$-path $g(\tilde{N})$ on Z^N is the range of the N-path $g(N)$. Let $\mathcal{G}$ denote the family of all full paths.

[3] Abusing notation, for $\lambda \in [0, X_N]$ we write $g(N,\lambda)$ instead of $g(N)(\lambda)$.

[4] Here, for a set $B \subseteq Z^{\tilde{N}}$, we denote by $\mathrm{proj}_N[B]$ the projection of B on Z^N.

Fixed-Path Rationing Method, $\phi^{(g^+,g^-)}$: Given two paths $g^+, g^- \in \mathcal{G}$, the fixed-path rationing method $\phi^{(g^+,g^-)}$ is defined as follows. For all $N \subseteq \mathcal{N}$ and all $(R,\Omega) \in \mathcal{E}^N$, (i) when $\Omega \leq \sum_{j\in N} p(R_j)$, there exists $\lambda \in Z$ such that for all $i \in N$, $\phi_i^{(g^+,g^-)}(R,\Omega) \equiv \min\{p(R_i), g_i^+(N,\lambda)\}$, and $\sum_{i\in N} \min\{p(R_i), g_i^+(N,\lambda)\} = \Omega$; and (ii) when $\Omega \geq \sum_{j\in N} p(R_j)$, there exists $\lambda \in Z$ such that for all $i \in N$, $\phi_i^{(g^+,g^-)}(R,\Omega) \equiv \max\{p(R_i), g_i^-(N,\lambda)\}$, and $\sum_{i\in N} \max\{p(R_i), g_i^-(N,\lambda)\} = \Omega$.[5]

Moulin (1999) characterized the class of fixed-path rationing methods by the following four axioms. First, a rule only selects efficient allocations.[6] Second, no agent can gain by misrepresenting his preference relation. Third, when some agents leave with their allotments, then the rule allocates the remaining amount to the agents who did not leave in the same way as before. Fourth, the amount assigned to each agent weakly increases whenever the collective endowment increases.[7]

Efficiency: For all $N \subseteq \mathcal{N}$ and all $(R,\Omega) \in \mathcal{E}^N$, if $\Omega \leq \sum_{i\in N} p(R_i)$, then $\varphi(R,\Omega) \leq p(R)$, and if $\Omega \geq \sum_{i\in N} p(R_i)$, then $\varphi(R,\Omega) \geq p(R)$.

Strategy-Proofness: For all $N \subseteq \mathcal{N}$, all $i \in N$, and all $(R,\Omega),(R',\Omega) \in \mathcal{E}^N$ such that $R_{N\setminus\{i\}} = R'_{N\setminus\{i\}}$, $\varphi_i(R,\Omega) R_i \varphi_i(R',\Omega)$.

Consistency: For all $N' \subseteq N \subseteq \mathcal{N}$, all $(R,\Omega) \in \mathcal{E}^N$, and all $i \in N'$, $\varphi_i(R_{N'}, \sum_{j\in N'} \varphi_j(R,\Omega)) = \varphi_i(R,\Omega)$.

Resource-Monotonicity: For all $N \subseteq \mathcal{N}$ and all $(R,\Omega),(R,\Omega') \in \mathcal{E}^N$, if $\Omega \leq \Omega'$, then $\varphi(R,\Omega) \leq \varphi(R,\Omega')$.

Theorem 2.1 (Moulin, 1999) *A rule satisfies efficiency, strategy-proofness, consistency, and resource-monotonicity if and only if it is a fixed-path rationing method.*

3 Proof of Sufficiency

Throughout let φ be a rule satisfying the properties of Theorem 2.1.

[5]Note that in (i) and (ii) λ is unique if $\sum_{i\in N} p(R_i) \neq \Omega$.

[6]Sprumont (1991) pointed out that *efficiency* is equivalent to same-sidedness. Below we use same-sidedness in defining *efficiency*.

[7]When it is unambiguous, we sometimes use $\leq$ and $\geq$ to denote the vector partial ordering.

Lemma 3.1 *φ satisfies peaks-onliness, i.e., for all $N \subseteq \mathcal{N}$ and all $(R,\Omega),(R',\Omega) \in \mathcal{E}^N$, if $p(R) = p(R')$, then $\varphi(R,\Omega) = \varphi(R',\Omega)$.*

Proof. Let $N \subseteq \mathcal{N}$, $i \in N$, and $(R,\Omega),(R',\Omega) \in \mathcal{E}^N$ be such that $p(R_i) = p(R'_i)$ and $R_{N\setminus\{i\}} = R'_{N\setminus\{i\}}$. By repeating the argument for profiles that differ only in one agent's preference, it suffices to show that $\varphi(R,\Omega) = \varphi(R',\Omega)$. By *efficiency* and *strategy-proofness*, $\varphi_i(R,\Omega) = \varphi_i(R',\Omega)$. Thus, $\sum_{j\in N\setminus\{i\}} \varphi_j(R,\Omega) = \sum_{j\in N\setminus\{i\}} \varphi_j(R',\Omega)$ and $R_{N\setminus\{i\}} = R'_{N\setminus\{i\}}$. Hence, by $|N| \in \{1,2\}$ or *consistency*, $\varphi(R,\Omega) = \varphi(R',\Omega)$. $\square$

Let $R^X \in \mathcal{R}^{\mathcal{N}}$ be such that for all $i \in \mathcal{N}$, $p(R_i^X) = X_i$. For all $N \subseteq \mathcal{N}$ and all $\lambda \in [0, X_N]$, let $g^+(N,\lambda) \equiv \varphi(R_N^X, \lambda)$. Let $g^+ \equiv (g^+(N))_{N\subseteq\mathcal{N}}$.

Let $R^0 \in \mathcal{R}^{\mathcal{N}}$ be such that for all $i \in \mathcal{N}$, $p(R_i^0) = 0$. For all $N \subseteq \mathcal{N}$ and all $\lambda \in [0, X_N]$, let $g^-(N,\lambda) \equiv \varphi(R_N^0, \lambda)$. Let $g^- \equiv (g^-(N))_{N\subseteq\mathcal{N}}$.

The following lemma applies to any two-agent population.

Lemma 3.2 *φ is a fixed-path method for $\{1,2\}$ with $\{1,2\}$-paths $g^+(\{1,2\})$ and $g^-(\{1,2\})$.*

Proof. We only prove the lemma for the case of excess demand. The case of excess supply is symmetric.

First, we show that $g^+(\{1,2\})$ is a $\{1,2\}$-path. Feasibility follows from the definition of φ, and monotonicity from *resource-monotonicity* of φ. Thus, $g^+(\{1,2\})$ satisfies (a) and (b). Finally, we show for all $(R,\Omega) \in \mathcal{E}^{\{1,2\}}$ such that $\Omega \leq p(R_1) + p(R_2)$, there exists $\lambda \in [0, X_{\{1,2\}}]$ such that

$$\varphi(R,\Omega) = (\min\{p(R_1), g_1^+(\{1,2\},\lambda)\}, \min\{p(R_2), g_2^+(\{1,2\},\lambda)\}). \qquad (1)$$

If $p(R) \geq \varphi(R^X_{\{1,2\}}, \Omega)$, then by *strategy-proofness*, $\varphi(R,\Omega) = \varphi(R^X_{\{1,2\}}, \Omega)$ and (1) holds for $\lambda = \Omega$. Without loss of generality, suppose that $p(R_1) < \varphi_1(R^X_{\{1,2\}}, \Omega)$. Then by *efficiency*, $\varphi_1((R_1, R_2^X), \Omega) \leq p(R_1)$. If $\varphi_1((R_1, R_2^X), \Omega) < p(R_1)$, then let $R'_1 \in \mathcal{R}^1$ be such that $p(R'_1) = p(R_1)$ and $\varphi_1(R^X_{\{1,2\}}, \Omega) P'_1 \varphi_1((R_1, R_2^X), \Omega)$. Since by *peaks-onliness*, $\varphi_1((R'_1, R_2^X), \Omega) = \varphi_1((R_1, R_2^X), \Omega)$, the previous relation contradicts *strategy-proofness*. Thus, $\varphi_1((R_1, R_2^X), \Omega) = p(R_1)$ and $\varphi_2((R_1, R_2^X), \Omega) \leq p(R_2)$. Hence, by *strategy-proofness*, $\varphi(R,\Omega) = \varphi((R_1, R_2^X), \Omega)$. Monotonicity of $g^+(\{1,2\})$

implies in the continuous model that $g_2^+(\{1,2\})$ is continuous with respect to λ.[8] Now in the continuous model (by the previous fact, $g_2^+(\{1,2\},0)=0$, $g_2^+(\{1,2\},X_{\{1,2\}})=X_2$, and the intermediate value theorem) and in the discrete model (by monotonicity of $g^+(\{1,2\})$), there exists $\lambda' \in [0, X_{\{1,2\}}]$ such that $g_2^+(\{1,2\},\lambda') = \varphi_2(R,\Omega)$. By monotonicity of $g^+(\{1,2\})$ and $\varphi_2(R,\Omega) \geq \varphi_2(R^X,\Omega)$, we have $\lambda' \geq \Omega$ and $g_1^+(\{1,2\},\lambda') \geq g_1^+(\{1,2\},\Omega) > p(R_1)$. Hence, (1) holds for $\lambda = \lambda'$. □

Lemma 3.3 *g^+ is a full path.*

Proof. It is easy to check that for all $N \subseteq \mathcal{N}$, $g^+(N)$ satisfies (a) and (b). Let $N \subseteq \tilde{N} \subseteq \mathcal{N}$. By *consistency* of φ and the definition of g^+, $\mathrm{proj}_N[\gamma(g^+(\tilde{N}))] \subseteq \gamma(g^+(N))$. Let $\lambda' \in [0, X_N]$. Monotonicity of $g^+(\tilde{N})$ implies in the continuous model that $\sum_{i\in N} g_i^+(\tilde{N})$ is continuous with respect to λ. Similarly to Lemma 3.2, then $\sum_{i\in N} g_i^+(\tilde{N},0)=0$, $\sum_{i\in N} g_i^+(\tilde{N},X_{\tilde{N}}) = X_N$, and monotonicity of $g^+(\tilde{N})$ imply that there exists $\lambda'' \in [0, X_{\tilde{N}}]$ such that $\sum_{i\in N} g_i^+(\tilde{N},\lambda'') = \lambda'$. Thus, by *consistency* of φ and the definition of g^+, we have for all $i \in N$ $g_i^+(\tilde{N},\lambda'') = \varphi_i(R^X_{\tilde{N}},\lambda'') = \varphi_i(R^X_N,\lambda') = g_i^+(N,\lambda')$. Thus, $g^+(N,\lambda') \in \mathrm{proj}_N[\gamma(g^+(\tilde{N}))]$ and $\mathrm{proj}_N[\gamma(g^+(\tilde{N}))] \supseteq \gamma(g^+(N))$. Hence, g^+ satisfies (c) and g^+ is a full path. □

Similarly it can be shown that g^- is a full path.

Lemma 3.4 $\varphi = \phi^{(g^+,g^-)}$.

Proof. We only prove the lemma for the case of excess demand. Suppose that there exist $N \subseteq \mathcal{N}$ and $(R,\Omega) \in \mathcal{E}^N$ such that $\Omega \leq \sum_{i\in N} p(R_i)$ and $\varphi(R,\Omega) \neq \phi^{(g^+,g^-)}(R,\Omega)$. Then there exist $i,j \in N$ such that

$$\varphi_i(R,\Omega) < \phi_i^{(g^+,g^-)}(R,\Omega) \text{ and } \varphi_j(R,\Omega) > \phi_j^{(g^+,g^-)}(R,\Omega). \tag{2}$$

Let $\Omega' \equiv \varphi_i(R,\Omega) + \varphi_j(R,\Omega)$ and $\Omega'' \equiv \phi_i^{(g^+,g^-)}(R,\Omega) + \phi_j^{(g^+,g^-)}(R,\Omega)$. By *consistency* of $\phi^{(g^+,g^-)}$, $\phi_i^{(g^+,g^-)}(R_{\{i,j\}},\Omega'') = \phi_i^{(g^+,g^-)}(R,\Omega)$ and $\phi_j^{(g^+,g^-)}(R_{\{i,j\}},\Omega'') = \phi_j^{(g^+,g^-)}(R,\Omega)$. Thus, by Lemma 3.2,

$$\varphi_i(R_{\{i,j\}},\Omega'') = \phi_i^{(g^+,g^-)}(R,\Omega) \text{ and } \varphi_j(R_{\{i,j\}},\Omega'') = \phi_j^{(g^+,g^-)}(R,\Omega). \tag{3}$$

[8] Thomson (1994b, Proof of Theorem 2, Part (i)) formally shows that if φ is *efficient* and *resource-monotonic*, then for all $N \subseteq \mathcal{N}$ and all $R \in \mathcal{R}^N$, $\varphi(R,\cdot)$ is continuous with respect to Ω (and therefore $g_2^+(\{1,2\})$ is continuous with respect to λ).

By *consistency* of φ, $\varphi_i(R_{\{i,j\}}, \Omega') = \varphi_i(R, \Omega)$ and $\varphi_j(R_{\{i,j\}}, \Omega') = \varphi_j(R, \Omega)$. Now the previous fact combined with (2) and (3) contradicts *resource-monotonicity* of φ. $\square$

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