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COLLUSION IN A MODEL OF REPEATED AUCTIONS

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RÉSUMÉ

Un modèle d'enchère au premier prix est formulé où les acheteurs se rencontrent d'une façon répétée en tirant des valeurs privées à chaque période de manière indépendante. Notre attention se porte sur les équilibres collusifs symétriques lorsque les transferts entre joueurs ne sont pas permis. Comme dans l'approche introduite par Abreu, Pearce et Stacchetti, les punitions sont endogénisées et nous les utilisons pour construire les stratégies de collusion. Cette analyse diffère d'une analyse standard des jeux répétés à cause de la présence d'information incomplète. Les stratégies collusives optimales sont en général inefficaces et possèdent une nature "bang-bang" qui implique la stabilité des cartels. Les réponses du vendeur sont également étudiées. Dans ce contexte de jeu répété, de nouveaux résultats qui ne sont pas présents dans un contexte statique apparaissent.

Mots clés : enchères, collusion, jeux répétés

ABSTRACT

A model of first price sealed bid auctions is developed where bidders meet repeatedly while independently drawing private valuations in each period. Attention is focused on symmetric collusive bidding equilibria when side-payments are not allowed. Via an approach introduced by Abreu, Pearce and Stacchetti, endogenous punishments are characterized and used in the construction of optimal collusive bidding strategies. This analysis differs from usual repeated game treatments due to the presence of incomplete information. Optimal collusive bidding strategies are generally inefficient and have a bang-bang nature which implies that defection is never observed. Auctioneer responses are also studied, which in this explicitly dynamic setting give rise to insights not apparent in a static formulation.

Key words : auctions, collusion, repeated games

1 Introduction

It is well recognized that auctions account for a significant share of current economic activity. From a pragmatic point of view, this is possibly why auctions have been the focus of so much research effort for nearly forty years. With few exceptions, the literature has maintained the assumption that agents participate in a one shot game: an auction is held and the participants part company never to meet again. This is the point of departure for this work, and is justified by the fact that in most auctions there is a core of long run bidders[1]. The repeated nature of such an auction immediately suggests the possibility of bidder collusion. Recently there has been some progress made into understanding the behavior of rational agents colluding in auctions, but this work has abstracted away from a repeated game analysis. For internal consistency this is reason enough to study collusion in an explicitly repeated environment. Studying collusion in auctions in a repeated setting is pertinent for several additional reasons. Firstly, the predictions generated by a static model do not always carry over to a dynamic model (in a way to be made clear below). Secondly, a dynamic model yields richer results than its static counterpart. And thirdly, an entire class of results on auctioneer behavior emerges which does not arise in a static model.

One of the earliest examinations of collusion in auctions was done by Comanor & Schankerman (1976) who examined rotating bid schemes (the most famous case-study is perhaps the "phases-of-the-moon scheme" detailed by Smith (1961)). They also studied a paradox in auction behavior: that of identical bids submitted by bidders[2]. McAfee & McMillan (1992) were able to resolve this paradox by showing that if bidders were unable to effectuate sidepayments amongst themselves, perhaps because of a "paper trail" which makes costly detection very likely, then the optimal response of a cartel would be for every bidder to bid the reserve price and let the auctioneer act as a randomization device. Of course if the auctioneer were to award the good in a predetermined way, say to the bidder who's name was first alphabetically, then a little bit more coordination by the ring is required. The key insight here is that a ring must renounce efficiency in order to overcome incentive compatibility problems brought about by the existence of privately held information. Though they only examined a static model, McAfee and McMillan assumed the existence of some sort of trigger-strategy which makes adherence to the cartel more profitable than defection.

When the restriction on sidepayments is lifted, the analysis changes rather drastically. As Graham & Marshall (1987) show, second price and English auctions are susceptible to a mechanism called a pre-auction knockout which not only succeeds in winning the object at the reserve price (when the cartel is all inclusive), but always awards the object to a buyer who values it the most. Efficiency is retained. In their paper, McAfee & McMillan (1992) also study "strong cartels". These cartels can effectuate transfers amongst themselves and can exclude non serious bidders. They also propose a mechanism which is efficient and wins the item at the reserve price. Again, these models abstract away from repeated play which must be used to justify obedience to such mechanisms.

Friedman (1971) was the first to formalize the "folk" theorem showing that repeated partnerships enable players to coordinate to equilibria which Pareto dominate any single stage Nash equilibrium. Usual analysis has abstracted away from any uncertainty, but if a model of auctions is to be studied one must keep in mind that uncertainty is the raison d'être for auctions. In fact it is this incomplete information in an auction which changes the analysis of repeated play, since players (in a sense) play a different game in each period. The perfect information assumption has been relaxed by Green & Porter (1984) as well as by Abreu, Pearce & Stacchetti (1986) and Abreu, Pearce & Stacchetti (1990), mainly through the study of Cournot oligopoly where each period's price gives an imperfect signal about (private information) firm-specific quantities. The uncertainty in auctions is somewhat different from the uncertainty described above. In Cournot competition an "auction-like" uncertainty would be more akin to firms holding private information about stochastic costs, as opposed to the demand side uncertainty studied by Green and Porter and Abreu, Pearce and Stacchetti. [3] Nevertheless, the approach introduced by Abreu, Pearce and Stacchetti proves to be particularly useful in studying repeated auctions.

An immediate outcome of studying explicitly repeated auctions is that one can easily give an example of a series of auctions where McAfee and McMillan's strategy of bidding the reserve price is not supportable. Thus, added to the familiar incentive compatibility constraints is a new dynamic "trigger" constraint (or in the terminology used by Abreu, Pearce and Stacchetti, "admissibility"). Roughly, the intuition is that a bidder can value an object very highly today– so much so that he is prepared to face any credible punishment in return for possessing the item today. The inefficiency of the McAfee and McMillan "flat" bidding function is the driving force behind this behavior. Section 2 gives a simple example of this phenomenon. The obvious question is that if for certain series of auctions bidding the reserve price is not an admissible scheme, then what is? and what is the optimal scheme? Furthermore, what can be said about cartel behavior across time: can incomplete information cause some kind of instability in the cartel? Does the role of the auctioneer differ from that of a static model where his main weapon in combatting collusion is the reserve price? This paper, therefore, is an attempt to answer the above questions.

Section 3 presents the model to be studied through a series of assumptions. Section 4 shows that the method of Abreu, Pearce and Stacchetti can be used in auction games. Some of the major points of the paper are stated in section 5 where optimal collusive bidder behavior is studied. The role of the auctioneer is presented in section 6 where several implications arise only due to the repeated nature of the scenario. Section 7 briefly concludes. Many of the proofs can be found in the appendix.

2 An Example

To motivate the following sections, this section presents a simple example. Consider two bidders who meet repeatedly in a first price sealed bid auction, discounting future earnings with a common discount rate $\delta \in (0,1)$, drawing independent valuations, denoted v, each period from a uniform distribution over $[0,1]$. Following McAfee and McMillan, suppose they decide to use a form of tacit collusion to increase their expected profits where each bids zero (the reserve price) in each period. The auctioneer is assumed to randomize equally in the case of a draw. Suppose that this collusion specifies that each player use Nash strategies forever if a winning bid above zero is ever observed. In order for any bidder type to prefer colluding it is necessary and sufficient that the highest valuation type prefer to adhere to cartel rules:

$$\frac{1}{2} + \frac{\delta}{1-\delta}\frac{1}{4} \geq 1 + \frac{\delta}{1-\delta}\int_0^1 v(1-v)dv.$$

The sufficiency follows from the fact that the incentive to cheat for a player with valuation one is greater than for any other valuation. The left hand side represents the gains to obeying the collusive rule for an agent with valuation one, and the right hand side represents the gains to defecting from the collusive rule for an agent with valuation one. However, for any $\delta < 6/7$ this collusive rule will eventually provoke defection (the admissibility constraint is not satisfied). The inefficiency of the collusive rule drives a high valuation agent to prefer cheating. The obvious question becomes what is the form of collusion which should be used to guarantee the participants the highest discounted expected payoffs? Can some kind of defection be permitted? How should defectors be punished?

3 Assumptions and Notation

A set $N = \{1, 2, \ldots, n\}$ of ex-ante identical potential buyers compete in an infinitely repeated auction game with discounting. Detailed assumptions can be found below.

The Stage Game

(Assumption 1). Each potential buyer is ex-ante symmetric, drawing an independent private value for the object in question from a common continuous distribution function $F(\cdot)$ with strictly positive continuously differentiable density, $f(\cdot)$, defined on a compact support $[0, a]$. For reasons to be made clear later assume that the hazard function, $H(v) := \frac{1-F(v)}{f(v)}$, is strictly decreasing in v.[4] Furthermore F and f are common knowledge.

(Assumption 2). Each potential buyer is risk neutral.

(Assumption 3). The auctioneer allocates objects via a first price auction, randomizing equally among the winners in the case of a draw. The auctioneer publicly announces the amount of the winning bid, but does not announce any other information (including the identity of the winner).

(Assumption 4). Sidepayments are not permitted between potential buyers.

(Assumption 5). The seller's reserve price is normalized to zero.

(Assumption 6). Unicity of equilibrium to the stage game is assumed. [5]

The Repeated Game

Let $\beta(t) : [0, a] \times [0, c] \times \overset{t-1 \text{ times}}{\ldots\ldots\ldots} \times [0, c] \to [0, c]$ be a bidding function at time t. This bidding function sends current valuation as well as all previous winning bids, $(b^w(t) \in [0, c])$ into the bidding space $[0, c]$. Let $\beta = \underset{t \in \mathbb{N}}{\times} \beta(t)$ be the strategy set of available bidding functions.

(Assumption 7). Players discount future stage profits with a common discount rate $\delta \in (0, 1)$.

(Assumption 8). Valuations are drawn independently and identically across time.

(Assumption 9). The cartel restricts itself to symmetric and undominated (in the sense of Pareto) collusive schemes.

(Assumption 10). The equilibrium concept used is that of Rubinstein's (1979) (subgame) perfect equilibrium.

Assumption 7 above permits the restriction that each potential buyer submit bids in a compact interval $[0, c]$. This can be done by taking c to be defined as the highest possible payoff one can receive in the entire game, i.e. $c = \frac{a}{1-\delta}$. Assumption 9 can be done away with by imposing the Nash bargaining axioms. This would not be a departure from the mainstream of the literature since modelers usually choose such a focal point when confronted with a continuum of possible payoff vectors. However, providing a criterium for the selection of equilibria in repeated games is outside the scope of this paper. Our objective is simply to study the best symmetric collusive strategy. Remark that under the informational structure imposed by Assumption 3 a cartel must, and can, be all encompassing. This is because a cartel cannot distinguish cheating by a member or non-member, and it can credibly menace any deviation with a punishment. Finally, note that due to the assumptions of ex-ante symmetry and continuous distribution function, a theorem from Milgrom & Weber (1982) can be used to show the existence of a symmetric bidding equilibrium where the bidding functions are increasing in valuation.

4 Static Representation of Repeated Auctions

With these assumptions and notations in mind, we now proceed to analyze the repeated game as a single stage game. Abreu et al. (1986) were the first to use a generalization of the techniques of dynamic programming to analyze noncooperative games. This permits a very convenient representation of repeated games as much more tractable single stage games. The intuition is the following. The single stage representation of a repeated game "factorizes" the gains accruing to each player into two parts. The first part is the gain from the current stage game and the second part is expected, future gains which can be a function of any observable action taken in the current period. So this representation needs two elements: a payoff function for the stage game and a function describing future gains. Proposition 1 states that an equilibrium to the single stage game which satisfies a "self generation" criterium is an equilibrium to the repeated game. Proposition 2 states that any perfect equilibrium to the repeated game can be represented as a Nash equilibrium to the single stage game. Thus the single stage representation is equivalent to the dynamic game. Before stating these propositions some definitions are presented.

Definition 1 *Let W be a bounded Borel subset of $\mathbb{R}$. Let $\beta : [0,a] \to [0,c]$ and $U : [0,c] \to W$ be Borel measurable functions and call (β, U) a collusive mechanism. (β, U) is called admissible with respect to W iff for all $b \notin \beta([0,a])$:*

$$\pi(v, \beta(v)) + \delta \mathbb{E}_{b^w} \left(U(b^w) \mid \beta(v) \right) \geq \pi(v, b) + \delta \mathbb{E}_{b^w} \left(U(b^w) \mid b \right)$$

$\pi(v, b)$ is the expected payoff of an agent with valuation v who bids b given that the other agents follow strategy β. $U(b^w)$ represents the expected gains of all agents, given that the winning bid is b^w. The operator $\mathbb{E}_{b^w}(\cdot \mid b)$ is the conditional expectation of future gains, from the perspective of an agent having bid b where all other players follow strategy β. Note that this conditional expectation is well defined due to the assumptions on (Borel) measurability of β and U, as well as the boundedness of W.

Furthermore define $u(v; \beta, U) := \pi(v, \beta(v)) + \delta \mathbb{E}_{b^w} \left(U(b^w) \mid \beta(v) \right)$. Note that an ex-ante value can be attached to $u(v; \beta, U)$, denoted as $\mathbb{E}_v u(v; \beta, U)$, by taking the expectation over the valuation. Finally a set valued function $B(W)$ is defined.

Definition 2 *For any (Borel) $W \subset \mathbb{R}$ define:*

$$B(W) := \{\mathbb{E}_v u(v; \beta, U) \mid (\beta, U) \text{ is admissible wrt } W\}$$

Definition 3 *A bounded (Borel) W such that $W \subset B(W)$ is called self generating.*

Propositions 1 and 2 are now corollaries to the propositions on self-generation and factorization in Abreu, Pearce and Stacchetti, since we are assured of the existence of the conditional expectation. The reader is referred to Abreu et al. (1986) for the proofs. In the following take V to be the set of perfect payoffs.

Proposition 1 *Take any bounded (Borel) $W \subset \mathbb{R}$ which is self generating. Then $B(W) \subset V$.*

Proposition 2 *$V = B(V)$.*

5 Optimal Collusive Rules

The goal of this section is to characterize optimal collusive schemes with the help of the Abreu, Pearce and Stacchetti static game. Results are presented in two subsections. Subsection 5.1 presents propositions which are used in Subsection 5.2 to present the problem of a maximizing cartel succinctly. Many of the proofs appear in the appendix

5.1 Punishments and Rewards

In explicitly repeated auctions there are two types of constraints which need to be satisfied. One corresponds to the usual self selection constraint: if β is the collusive bidding rule, then type v prefers bidding $\beta(v)$ to bidding $\beta(v')$. The other constraint will be referred to as admissibility. This constraint states that if β is the bidding rule, type v prefers bidding in $\beta([0,a])$, as opposed to bidding outside $\beta([0,a])$. These two constraints are qualitatively different because in the latter case, a defection can be detected.

Typically the incentive compatibility constraints force the mechanisms to be structured in such a way that types self select. In the problem at hand, the mechanism must indeed be so structured since valuations are not observable. Consider a collusive mechanism $\{\beta(\cdot), U(\cdot)\}$ where β is the bidding function mapping types into bids and where $U(b^w)$ is the future expected payoffs when the winning bid is b^w. Let $Q(v;\beta)$ denote the probability that a player bidding $\beta(v)$ wins the auction given that β is the bidding function used by everyone else. Since the auctioneer awards the object to the highest bidder, and in the case of draws randomizes, $Q(v;\beta)$ can be written explicitly. Obviously if v is in an interval where β suggests submitting an increasing bid, then $Q(v;\beta) = F(v)^{n-1}$. Otherwise if β is constant within the interval $[v_i, v_j]$ then for all $v \in [v_i, v_j]$ we have:

$$Q(v;\beta) = \frac{F(v_j)^n - F(v_i)^n}{n(F(v_j) - F(v_i))} \equiv Q(v_i, v_j).$$

Incentive compatibility can thus be written:

$$\begin{aligned} u(v;\beta,U) &:= [v - \beta(v)]Q(v;\beta) + \delta \mathbf{E}_{b^w}(U(b^w) \mid \beta(v)) \\ &\geq [v - \beta(v')]Q(v';\beta) + \delta \mathbf{E}_{b^w}(U(b^w) \mid \beta(v')) \quad \forall v' \in [0,a]. \end{aligned}$$

Where

$$\mathbf{E}_{b^w}(U(b^w) \mid \beta(v)) = \left[\int_v^a U(\beta(s)) dF(s)^{n-1} + U(b)F(v)^{n-1} \right].$$

Using the logic of Myerson (1981) we know that incentive compatibility is equivalent to demanding that

$$\frac{d}{dv}u(v;\beta,U) = Q(v;\beta)$$
$$Q(v;\beta) \text{ is nondecreasing in } v.$$

We can thus obtain a more convenient characterization of incentive compatibility which lets us explicitly solve for the bidding function in terms of type conditional probabilities of winning:

$$\beta(v) = v - \left[\frac{\int_0^v Q(s;\beta)ds + \delta\int_0^v [U(\beta(s)) - U(\beta(v))]dF(s)^{n-1}}{Q(v;\beta)}\right] \tag{1}$$

It is important to note that the bidding function will generally be composed of (positively) sloped and flat regions. From McAfee & McMillan (1992) we know that a perfectly flat bidding scheme is a "folk" type result which speaks to the case when δ is arbitrarily close to unity.

If we are to study collusion in auctions from an explicitly repeated point of view, as was suggested in the Introduction and in Section 2, another constraint must be imposed: admissibility. The admissibility constraints will be relevant whenever β contains flat regions. The following lemma furnishes some preliminary results. Its proof can be found in the appendix.

Lemma 1 *Let $\underline{U}$ denote the lowest expected profit credibly attainable. An incentive compatible mechanism $\{\beta(\cdot), U(\cdot)\}$ is admissible if and only if for all v^* at the highest edge of a flat bidding range we have:*

$$[v^* - \beta(v^*)]Q(v^*;\beta) + \delta U(\beta(v^*))F(v^*)^{n-1} + \delta\int_{v^*}^{a} U(\beta(s))dF(s)^{n-1} \geq$$
$$[v^* - \beta(v^*)]F(v^*)^{n-1} + \delta\underline{U}F(v^*)^{n-1} + \delta\int_{v^*}^{a} U(\beta(s))dF(s)^{n-1}.$$

The above is similar to the example introduced in Section 2, in that we compare gains to deviation with gains to compliance. The proof consists primarily of showing the existence of discontinuities in the bidding function, showing that the imposition of the harshest possible punishment is always beneficial. Finally we prove that if all v^* at the highest edge of a flat bidding range respect admissibility then all types do.

Using the incentive compatibility constraint (1) and the above lemma, we can rewrite admissibility as:

$$\delta[U(\beta(v^*)) - \underline{U}] \geq$$
$$\left[\int_0^{v^*} Q(s;\beta)ds + \delta \int_0^{v^*} [U(\beta(s)) - U(\beta(v^*))]dF(s)^{n-1}\right] \frac{F(v^*)^{n-1} - Q(v^*;\beta)}{F(v^*)^{n-1}Q(v^*;\beta)}. \quad (2)$$

Note that when the bidding function is strictly increasing admissibility is automatically satisfied.

Define $\overline{\mathcal{M}}$ as an incentive compatible and admissible mechanism which yields the highest level of expected utility $\overline{U}$. Define $\underline{\mathcal{M}}$ as an incentive compatible and admissible mechanism which yields the lowest level of expected utility $\overline{U}$. From section 5 of Abreu et al. (1986) the set of perfect payoffs is compact, so we are assured that $\overline{\mathcal{M}}$ and $\underline{\mathcal{M}}$ are well defined. The remainder of this section aims to characterize $\overline{\mathcal{M}}$ and $\underline{\mathcal{M}}$.

In order to satisfy admissibility, credible punishments must be available. Proposition 3 shows that the harshest possible punishment gives the same payoffs as the single stage Nash equilibrium. Its proof can be found in the appendix.

Proposition 3 *No perfect strategy can give less than the payoff associated with the stage game Nash equilibrium repeated ad infinitum.*

This proposition is an important step in endogenizing punishments as it characterizes the gains to optimal (in the sense of Abreu (1986)) punishments. The proof relies heavily on the fact that $H(v)$ is decreasing in v along with the fact that the only information released by the auctioneer is the winning bid (Assumption 3).

Inequality (2) leads us to an important result on collusion in repeated auctions.

Proposition 4 *An optimal collusive mechanism, $\overline{\mathcal{M}} = \{\beta(\cdot), U(\cdot)\}$, must exhibit the bang-bang property:*

$$U(b) = \overline{U} \qquad \textit{for almost every } b \in \beta([0, a]).$$

Proof of Proposition 4 Let $\{Q(\cdot;\beta), U(\cdot)\}$ be an optimal collusive mechanism, generating $\overline{U}$, for which the bang-bang property does not hold. Therefore, there exists an interval, $[v_1, v_2]$, such that $U(\beta(v)) < \overline{U}$ for all $v \in [v_1, v_2]$. In this case we show that there exists an alternative incentive compatible, admissible mechanism which generates higher rents for all types. This mechanism consists of raising $U(\beta(v))$ by some small amount ΔU whenever $v \in [v_1, v_2]$. Denote this new continuation function by U^*. All flat bidding regions above v_2 are cut into two regions otherwise, the probability of winning is held constant for all other types.

From equation (1), if we increase continuation payoffs in $[v_1, v_2]$ and wish to preserve type conditional probabilities of winning an auction ($Q(\cdot, \beta)$), it is necessary to change bids in order to retain incentive compatibility. Bids of types lower than v_1 are unaffected by such a change. Studying inequality (2) immediately shows that types in $[v_1, v_2]$ will satisfy admissibility. The variation in the rents to $v < v_2$ are given by:

$$\Delta u(v_2) = [F(v_2)^{n-1} - F(v_1)^{n-1}]\Delta U$$

which is strictly positive for all $\Delta U > 0$.

Consider the intervals $[\underline{v}_i, \overline{v}_i]$ where $\underline{v}_i \geq v_2$ and for which the bidding function is flat. Find recursively a $\hat{v}_i$ where the following holds:

$$(\hat{v}_i - \underline{v}_i)[Q(\underline{v}_i, \overline{v}_i) - Q(\underline{v}_i, \hat{v}_i)] = \Delta u(\underline{v}_i) = \Delta u(\overline{v}_{i-i}).$$

Create a new incentive compatible bidding function (using the new continuation function U^*), β^* different from β only in that $Q(\cdot;\beta^*)$ exhibits discontinuities at all $\hat{v}_i$ but constant on all the $[\underline{v}_i, \hat{v}_i]$ and $[\hat{v}_i, \overline{v}_i]$.

We now have $\Delta u(v) = (v - \hat{v}_i)[Q(\hat{v}_i, \overline{v}_i) - Q(\underline{v}_i, \overline{v}_i)] \geq 0$ for all $v \in [\hat{v}_i, \overline{v}_i]$. Additionally for all $v \in [\underline{v}_i, \hat{v}_i]$ we have $\Delta u(v) = [v - \underline{v}_i][Q(\underline{v}_i, \hat{v}_i) - Q(\underline{v}_i, \overline{v}_i)] + \Delta u(\underline{v}_i) \geq 0$. Hence the variation in rents for all types is non negative by construction. It remains to verify that the new mechanism is admissible for all $v > v_2$ if the original mechanism was.

First consider checking admissibility in $[\hat{v}_i, \overline{v}_i]$. With the original mechanism we have

$$\overline{v}_i - \beta(\overline{v}_i) = (\overline{v}_i - \hat{v}_i) + \frac{u(\hat{v}_i;\beta, U)}{Q(\underline{v}_i, \overline{v}_i)}.$$

Under the new mechanism we have that:

$$\overline{v}_i - \beta^*(\overline{v}_i) = (\overline{v}_i - \hat{v}_i) + \frac{u(\hat{v}_i;\beta^*, U^*)}{Q(\hat{v}_i, \overline{v}_i)}.$$

Since $u(\hat{v}_i;\beta^*, U^*) = U(\hat{v}_i;\beta, U)$, and $Q(\underline{v}_i, \overline{v}_i) < Q(\overline{v}_i, \hat{v}_i)$ we have that:

$$\overline{v}_i - \beta^*(\overline{v}_i) < \overline{v}_i - \beta(\overline{v}_i)$$

So the admissibility constraint, which can be written:

$$\delta[\overline{U} - \underline{U}]F(\overline{v}_i)^{n-1} \geq (\overline{v}_i - \beta^*(\overline{v}_i))[F(\overline{v}_i)^{n-1} - Q(\overline{v}_i;\beta^*, U^*)],$$

is sure to be satisfied, since the original mechanism was assumed to be admissible and $Q(\overline{v}_i;\beta^*,U^*) > Q(\overline{v}_i,\beta,U)$ and $(\overline{v}_i - \beta^*(\overline{v}_i)) < (\overline{v}_i - \beta(\overline{v}_i))$.

Finally, check admissibility in $[\underline{v}_i, \hat{v}_i]$. Notice that as ΔU approaches zero, $\hat{v}_i$ must approach $\underline{v}_i$. It follows that for a ΔU sufficiently close to zero, the admissibility constraint will be respected for $\hat{v}_i$. $\square$

Consider the McAfee and McMillan flat bidding scheme. In this instance, the incentive compatibility constraint is trivial since there is only one "advised" bid (the reservation price). However their static formulation abstracted away from admissibility as was alluded to in the example of section 2. Proposition 4 states that an optimal collusive scheme should be structured so that any bid in $\beta([0,a])$ be treated as respecting cartel rules, and any bid not in $\beta([0,a])$ should be treated as defection[6].

It follows directly from Proposition 4 that the bidding function takes on the familiar form:

$$\beta(v) = v - \frac{\int_0^v Q(s;\beta)ds}{Q(v;\beta)}. \tag{3}$$

It is also easy to verify that any flat section in the bidding function must be preceded and followed by discontinuities. Suppose that a flat section somewhere in the bidding function starts at valuation $v^* > 0$ and ends at valuation $v^{**} < a$. Since the probability of a bidder with valuation v^* winning is discretely greater than the probability of a bidder with valuation $v^* - \varepsilon$ for any $\varepsilon > 0$, the denominator of the above equation increases discretely at v^*. Therefore $\beta(v^*)$ must increase discretely as well. The argument that a flat section must be followed by a discontinuity is precisely the same.

5.2 Collusive Schemes as Optimization Problems

Define the operator Ψ in the following manner:

$$\Psi(K) = \sup_{\beta} \int_0^a H(v)[Q(v;\beta) - F(v)^{n-1}]dF(v) \tag{4}$$

subject to:

$$F(v)^{n-1}K \geq \frac{\int_0^v Q(s;\beta)ds}{Q(v;\beta)}[F(v)^{n-1} - Q(v;\beta)] \quad \forall v \in [0,a]. \tag{5}$$

The constraint is the admissibility constraint where the bid has been substituted out using equation (3) and where the variable K can be seen to take the place of $\delta[\overline{U} - \underline{U}]$. $\Psi(K)$ in equation (4) represents the difference in the per period rents between collusion using bidding function β and playing the Nash stage game equilibrium strategy. Note that if

$$\frac{\delta}{1-\delta}\Psi(K) \geq K$$

then the collusive scheme giving payoffs of $\Psi(K)$ every period is admissible. This amounts to comparing K with the future payoffs generated by K.

The following lemma regroups several preliminary results. Its proof is relegated to the appendix.

Lemma 2 *i) $\Psi(\cdot)$ is non decreasing in its argument.*

ii) $\Psi(0) = 0$.

iii) For any $K \geq a[1-(1/n)]$, $\Psi(K) = \int_0^a H(v)[\frac{1}{n} - F(v)^{n-1}]dF(v)$.

iv) There is a finite θ such that $\Psi(K) \leq \theta K$ for all K.

The above lemma contains some immediately interpretable results. The fact that $\Psi(\cdot)$ is non decreasing in its argument, means that gains to collusion are non decreasing in δ. And in fact, when players are perfectly myopic ($\delta = 0$) then collusion gains and the gains to the Nash stage game equilibrium strategy are equal. This of course follows from the fact that $\Psi(0) = 0$. The interpretation of part *iii* is straightforward to interpret and is best seen as a result of McAfee & McMillan (1992) who only consider the incentive compatibility constraint. The interpretation of condition *iv* will be postponed until the statement of proposition 5.

It is possible to prove that $\Psi(K)$ subject to constraint (5) is continuous by appealing to the Theorem of the Maximum, however this does not imply that collusive profits are continuous in δ. Consider Figures 1 and 2 which plot $\Psi(K)$ against K. The maximal admissible and incentive compatible collusive payoffs are given by the intersection of $\Psi(K)$ and the line with slope $\frac{1-\delta}{\delta}$. This follows because whenever $\Psi(K) \geq \frac{1-\delta}{\delta}K$ we know that the future expected collusive rents are large enough to satisfy admissibility required to enforce this collusion. As in Figure 1, if $\Psi(K)$ is concave in K, then collusive gains are indeed continuous in K. However if $\Psi(K)$ is not globally concave, as is the case in Figure 2 then a small change in δ could generate a large change in the intersection of $\Psi(K)$ with $\frac{1-\delta}{\delta}$. The relationship between the concavity of $\Psi(K)$ and the parameters of the problem is a complex one and we have no reason to believe that $\Psi(K)$ is globally concave for all distributions satisfying Assumption 1. Nevertheless we can state the following proposition.

Proposition 5 *i) For all δ there exists an optimal collusive scheme.*

ii) There exists a $\hat{\delta} > 0$ such that if $\delta < \hat{\delta}$ no collusion is possible.

Proof of Proposition 5 i) Let $\Psi^* = \sup_K \{\sup_\beta \int_0^a H(v)[Q(v;\beta) - F(v)^{n-1}]dF(v)\}$ subject to constraint (5). Let K^* and β^* be the arguments which obtain the sup. Assume that Ψ^* is not admissible, i.e. $\frac{\delta}{1-\delta}\Psi(K^*) < K^*$. But there exists some sequence $K_n \to K^*$ such that $\frac{\delta}{1-\delta}\Psi(K_n) \geq K_n$ for all n. If the sequence $\{K_n\}$ is non monotone or monotone decreasing, then we have a contradiction since $\Psi(K)$ is non decreasing in K. So assume that $K_n \uparrow K^*$. Since $K_n \uparrow K^*$, there exists a sequence $\varepsilon_n \downarrow 0$ such that $K_n + \varepsilon_n = K^*$. Making a substitution we obtain $\frac{\delta}{1-\delta}\Psi(K^*) < K_n + \varepsilon_n$. But for n sufficiently large, we have $\frac{\delta}{1-\delta}\Psi(K^*) < K_n$ which is a contradiction since $K^* \geq K_n$ and Ψ is nondecreasing.

ii) From Lemma 2 we know that there exists a finite number θ such that $\Psi(K) \leq \theta K$ for all K. For a collusive scheme to be admissible and incentive compatible we require that $\frac{\delta}{1-\delta}\Psi(K) \geq K$. Putting $\frac{1}{\theta} = \frac{\hat{\delta}}{1-\hat{\delta}}$ assures us that for all $\delta < \hat{\delta}$, this cannot be the case.

The next proposition narrows down the class of optimal collusive bidding functions.

Proposition 6 *If $\overline{U} > \underline{U}$ then an optimal collusive bidding function can contain no continuous increase in the bidding function.*

The proof of proposition 6 can be found in the appendix. The implications of this proposition are strong as it implies that an optimally colluding cartel will use a bidding function which has a finite range. For example all bidders with type in $[v_0, v_1]$ bid b_0, all bidders with type in $[v_1, v_2]$ bid b_1 etc. Athey, Bagwell & Sanchirico (1998) find a similar result for collusion in a Cournot oligopoly with incomplete information about cost.

6 Auctioneer Behavior

The tractability of the model used up to now has come at the expense of many simplifying assumptions. This current section argues that despite the simplicity of the model, analysis of collusion in an explicitly repeated environment can lead to interesting and robust insights as to how an auctioneer can best structure a series of auctions to resist bidder collusion. This section has three subsections, each presenting a different way that auctioneers could design auctions to lessen their losses due to collusion. The following lemma is a preliminary result used in each of the following subsections and its proof can be found in the appendix.

Lemma 3 *If an optimal collusive bidding function exhibits one or more discontinuities, then there exists at least one type indifferent between adhering to and defecting from prescribed behavior.*

6.1 Reserve Prices and Bidding Ceilings

The use of reservation prices to combat collusion has been studied by Graham & Marshall (1987) as well as McAfee & McMillan (1992). This analysis is strengthened by the observation that auctioneers in fact do use reserve prices to increase their profits in the face of collusion.[7] In McAfee & McMillan (1992), since the cartel colludes so that all types are awarded the good with equal probability, the objective of the auctioneer is to solve the following:

$$\max_{r}: \ (r - v_0)[1 - F(r)^n],$$

where r denotes the reserve price and v_0 the seller's valuation.

In the more general case studied in this paper, the task of finding an optimal reserve price is a more daunting one. This is principally due to the fact that an optimal collusive rule depends on the reservation price in a non trivial way (as was the case studied by McAfee & McMillan (1992)). The objective of the auctioneer is:

$$\max_{r}: \int_{r}^{a} \beta(v, r) dF(v)^n$$

where the bidding function not only depends on the valuation but also on the reserve price. It is an easy exercise to extend the analysis of subsection 5.2 to accommodate a non zero reserve price. This is necessary to do in order to derive an optimal collusive bidding function for a given reserve price. Assuming differentiability and concavity of the problem, we are looking for r^* such that:

$$-\beta(r^*, r^*)nf(r^*)F(r^*)^{n-1} + \int_{r^*}^{a} \frac{\partial \beta}{\partial r}(v, r^*)dF(v)^n = 0.$$

In McAfee & McMillan (1992), $\frac{\partial\beta}{\partial r}(v,r)$ is equal to one. In a more traditional non collusive setting (for example Myerson (1981)) $\frac{\partial\beta}{\partial r}(v,r)$ is equal to $\frac{F(r)^{n-1}}{F(v)^{n-1}}$. In a general collusive setting, it is not possible to assign a value to $\frac{\partial\beta}{\partial r}(v,r)$. This is due to the ambiguous effect of a reserve price on the admissibility constraint: a change in the reserve price affects the admissibility constraint through $\underline{U}$; additionally a change in the reserve price modifies a collusive bidding function by changing type conditional probabilities of winning the auction. Hence, the impact of the reserve price on the ability of the cartel to collude is ambiguous, a fortiori so is its impact on the equilibrium bidding schedule.

A tool to increase auctioneer profits which is only apparent from an explicitly repeated context is a bidding ceiling. Consider the example presented in section 2 but with a bidding ceiling of 1/4. Single stage expected Nash profits now increase and become 31/162. Therefore in order for there to be no type of bidder who does not want to defect from the strategy of bidding zero it is necessary and sufficient that:

$$\frac{1}{2} + \frac{\delta}{1-\delta}\frac{1}{4} \geq 1 + \frac{\delta}{1-\delta}\frac{31}{162}$$

Where the left hand side represents the gains to obeying the collusive rule for an agent with valuation one, and the right hand side represents the gains to defecting from the collusive rule for an agent with valuation one. This equation is only satisfied for δ larger than 6/7. That bidding ceilings can positively effect auctioneer profits is perhaps surprising, but the intuition is quite straightforward if one thinks in terms of a repeated context: lowering the bidding ceiling makes future punishments less severe, and a less severe punishment can support a less profitable form of collusion. The following proposition gives some general conditions as to when lowering a bidding ceiling can be useful.

Proposition 7 *Suppose that the current price ceiling, $\overline{b}$, is strictly greater than the highest bid prescribed by the optimal collusive bidding function. Then auctioneer profits can be strictly increased by a lowering of the bid ceiling.*

Proof of Proposition 7 Assume, without loss of generality, that the bidding ceiling is less than or equal to the highest single stage Nash bid. Now remark that since $\overline{b}$ is strictly greater than the highest prescribed bid, $\overline{b}$ can be lowered by $\varepsilon > 0$ while still preserving this inequality. Furthermore since lowering $\overline{b}$ by any amount increases the single-stage Nash expected payoffs, any indifferent types will now strictly prefer defecting from the cartel to obedience. Therefore, the cartel must remedy this problem by decreasing the length of certain "flat spots" on the bidding function or adding more discontinuities. Either response results in lower profits to the cartel and higher profit to the auctioneer. If the optimal bidding function contains no discontinuities, then there may not be an indifferent agent (lemma 3). However, by choosing the bidding ceiling such that the Nash single stage payoff and the collusive (bidding zero) single stage payoff differ by less than $a\frac{n-1}{n}\frac{1-\delta}{\delta}$, then at least one type strictly prefers defection. This new bidding ceiling obviously leads to a higher average winning bid. $\square$

Stated differently, the above says that a necessary condition for an auctioneer to maximize profits is to have the bidding ceiling equal to the highest prescribed bid of an optimal collusive bidding function. This is a strong result since it implies that auctioneers always profit from the imposition of a ceiling. Unfortunately, while bidding ceilings and reserve prices may be useful in increasing auctioneer payoffs the following proposition shows that they are not sufficient for maximizing auctioneer profits. The intuition is similar to that of proposition 7 in that one can outlaw other portions of the range of bids in order to weaken future punishments.

Proposition 8 *Controlling the bidding ceiling and reserve price is not sufficient for an auctioneer to maximize profit.*

Proof of Proposition 8 Suppose that the auctioneer has maximized his profit with respect to the bidding ceiling and floor.[8] From lemma 3 and proposition 7 there exists at least one type indifferent between defecting from and adhering to cartel rules. Denote the prescribed bids of this bidding function $\{b_1, \ldots, b_k\}$, such that $b_i > b_{i-1}$ for all i. Proposition 7 implies that $\bar{b} = b_k$. Now outlaw bidding in an interval just below b_k: $(b_k - \varepsilon, b_k)$. Single stage Nash payoffs must increase when any interior portion of the bidding range is outlawed. Since there exists a type indifferent between adhering and defecting under the old system, this type now strictly prefers defecting. Therefore, in order for cartel profits not to decrease, the cartel must come up with a more profitable collusive scheme. By assuming that the cartel had chosen an optimal collusive bidding function, the only way for the cartel to improve payoffs is to permit bidding at $b_k - \varepsilon$. The key is recognizing that since ε can be taken to be arbitrarily small the admissibility constraint will not be binding between bidding $b_k - \varepsilon$ and b_k. Consider three possibilities: 1) replace b_k by $b_k - \varepsilon$, 2) replace b_{k-1} by $b_k - \varepsilon$, 3) add a new bidding level at $b_k - \varepsilon$. In each case if a cartel were able to increase its profitability this would imply that it had originally chosen a non optimal collusive bidding function–a contradiction. $\square$

The surprising results on bidding ceilings is quite intuitive, yet to our knowledge bidding ceilings do not exist.

6.2 Tiebreaking Rules

The inefficiency of the McAfee and McMillan collusive scheme has been retained to a certain extent; thus one would expect to observe identical bids. Due to the presence of these identical bids the choice of tiebreaking rule used by the auctioneer becomes important through its effect on the admissibility constraint. Consider the example in section 2. Assuming a randomization on the part of the auctioneer in case of ties (ex post randomization) we see that the McAfee and McMillan scheme is enforceable only when $\delta \geq 6/7$. Now consider what happens if the auctioneer, prior to bids being submitted, publicly designates a winner in case of equality in the highest bid (ex ante randomization). Assume the designation is made by a flip of a coin. In this case the necessary and sufficient condition for obedience to this scheme is the following:

$$0 + \frac{\delta}{1-\delta}\frac{1}{4} \geq 1 + \frac{\delta}{1-\delta}\int_0^1 v(1-v)dv.$$

The difference here is the first term on the left hand side. This term reflects that a player with a valuation of one has been designated as the loser in case of a tie. The gains to defecting from cartel rules have increased as is reflected by solving for δ in the above inequality to reveal $\delta \geq 12/13$. The above argument is formalized in the following proposition.

Proposition 9 *Bidder profits under ex post randomization are always at least as large as under ex ante randomization.*

Proof of Proposition 9 Consider the optimal collusive scheme $\{v_1^*, v_2^*, \dots\}$, and suppose $v \in [v_k^*, v_{k+1}^*]$. The admissibility constraint under ex post randomization is:

$$vQ(v_{k+1}^*, v_k^*) + \frac{\delta}{1-\delta} \sum_i Q(v_{i+1}^*, v_i^*) \int_{v_i^*}^{v_{i+1}^*} H(v)dF(v) \geq$$
$$vF(v_{k+1}^*)^{n-1} + \frac{\delta}{1-\delta} \int_0^a H(v)F(v)^{n-1}dF(v).$$

Now consider the admissibility constraint under ex ante randomization:

$$vF(v_k^*)^{n-1} + \frac{\delta}{1-\delta} \sum_i Q(v_{i+1}^*, v_i^*) \int_{v_i^*}^{v_{i+1}^*} H(v)dF(v) \geq$$
$$vF(v_{k+1}^*)^{n-1} + \frac{\delta}{1-\delta} \int_0^a H(v)F(v)^{n-1}dF(v).$$

Comparing $Q(v_{k+1}^*, v_k^*)$ with $F(v_k^*)^{n-1}$ implies that one can use lemma 3 to arrive at the following conclusions. If there exists one or more discontinuities in the bidding function then ex ante randomization strictly dominates ex post randomization. If there exists no discontinuity then ex ante randomization weakly dominates ex post randomization. □

The above argument implies that a cartel would always prefer to face ex post randomization. The existence of rotating bidding schemes may be interpreted to be the cartel's reaction to an "unaccommodating" auctioneer. That the effect on auctioneer profits of such a simple change in the randomization rule is unambiguous, it is surprising that one does observe auctioneers who use ex post randomization.

6.3 Bundling

Oftentimes services can be procured through auctions occuring with a regular frequency. One might think of a garbage collection contract being awarded by a municipality through auction every year. For the present purposes, the important part of this scenario is that the auction occurs every year. Once again consider the example of section 2, but now assume that the auctioneer sells 2 items at every other period–the auctioneer bundles items. In this case a necessary and sufficient condition for obedience to the McAfee and McMillan scheme is the following inequality:

$$2\frac{1}{2} + \frac{\delta^2}{1-\delta^2}\frac{1}{4} \geq 2 + \frac{\delta^2}{1-\delta^2}\int_0^1 v(1-v)dv.$$

Solving for δ one obtains $\delta \geq \sqrt{12/13}$. One observes two effects here. The first is the reduction in the effective discount rate from δ to δ^2. The second is the increase in the gains to cheating. Again we formalize the above intuition into the following proposition.

Proposition 10 *Bidder profits are always decreasing in the number of items bundled together.*

Proof of Proposition 10 For simplicity consider bundling two items together, and compare the corresponding admissibility constraint with the admissibility constraint when there is no bundling. Again consider the collusive scheme $\{v_1^*, v_2^*, \dots\}$, and suppose $v \in [v_k^*, v_{k+1}^*]$. One can write down the admissibility constraints and compare $2v[F(v_{k+1}^*)^{n-1} - Q(v_{k+1}^*, v_k^*]$ and $v[F(v_{k+1}^*)^{n-1} - Q(v_{k+1}^*, v_k^*)]$. The effect of the decrease in the discount factor is straightforward. Using lemma 3 one can reason along the same lines as in the previous proposition. If the bidding function has at least one discontinuity then domination by bundling is strict. If the bidding function contains no discontinuity then the domination is weak. □

It is important to stress that this last argument has ignored important considerations such as any costs a municipality may incur in making a long term commitment to a single service supplier.

7 Conclusion

Collusion without sidepayments in explicitly repeated auctions has been studied. It is the repeated nature of the model which differentiates it from earlier work. Techniques first developed by Abreu, Pearce and Stacchetti were used to endogenize punishments. These punishments differ from the usual trigger strategies in that an auction is a game of incomplete information. Collusion is characterized by its stability and inefficiency. Collusion is effectuated by submitting identical bids, or using a rotating bidding scheme–predictions which are commonly used cartel tactics. Studying an explicitly repeated game leads to several insights about optimal auctioneer behavior not apparent when studying a static game. The main finding of the paper, that cartels are obliged to use a randomization rule to overcome incentive compatibility problems, is supported by the observation that identical bids and rotating bid schemes are observed.

A Appendix

Proof of Lemma 1 Suppose β to be constant on $[v_i, v_j]$. First remark that $Q(v;\beta)$ is discontinuous at v_j, and that in particular $Q(v_j;\beta) < F(v_j)^{n-1}$. Suppose that the condition in the statement of the lemma is violated and that β is continuous at v_j. Then since $U(\beta(v_j+\varepsilon)) \geq \underline{U}$ for all $\varepsilon > 0$, there exists a small ε such that bidders of type v_j prefer to bid $\beta(v_j+\varepsilon)$, violating the incentive compatibility constraints. Now suppose that β is discontinuous at v_j. From equation (1) one can verify that β is also discontinuous at v_j. Therefore there exists a $b \notin \beta([0,a])$ which is arbitrarily close to $\beta([v_i, v_j])$ yet which discretely increases the probability of winning. A (credible) punishment must be available to dissuade such a deviation. This punishment can be made as severe as possible without harming cartel gains, in so far as it is only used off the equilibrium path, i.e. $b^w \notin \beta([0,a])$.

To prove the only if part, notice that admissibility implies that all types respect the above inequality, and in particular v_j respects the above inequality.

To prove the if part, define $\beta^+ = \lim_{v \downarrow v_j} \beta(v)$. Similarly define $\beta^- = \lim_{v \uparrow v_j} \beta(v)$. Since β is discontinuous at v_j we have that $\beta^+ > \beta^-$. Any bid in $(\beta^-, \beta^+]$ will win the auction with probability $F(v_j)^{n-1}$. The inequality in the statement of the lemma implies that v_j prefers following his equilibrium strategy rather than deviating. Fix $v < v_j$. By monotonicity of preferences, v also prefers bidding β^- than bidding in $(\beta^-, \beta^+]$. A similar argument shows that all $v > v_j$ prefer bidding β^+ to bidding in $(\beta^-, \beta^+]$. $\square$

Proof of Proposition 3 Notice first that Assumption 3 implies that punishments cannot be bidder specific–all players must receive the same expected payoff from punishment.

Gains to a player with valuation v from the collusive mechanism (β, U) are:

$$\int_0^v Q(s;\beta)ds + \delta \int_0^a U(\beta(v))dF(v)^{n-1},$$

which can be used to calculate ex-ante rents:

$$\int_0^a H(v)Q(v;\beta)dF(v) + \delta \int_0^a U(\beta(v))dF(v)^{n-1}. \tag{6}$$

Consider infinite, or unrelenting, punishments. Let (β_1, U_1) be an incentive compatible collusive mechanism which minimizes (6):

$$\underset{\beta_1, U_1}{min}: \int_0^a H(v)Q(v;\beta_1)dF(v) + \delta \int_0^a U_1(\beta_1(v))dF(v)^{n-1}$$

Since payoffs are constrained to be in V, the perfect equilibrium set, and $V =$

$B(V)$, there exists another collusive couple, (β_2, U_2) giving the same payoffs as $\int_0^a [\mathbb{E}_b(U_1(b) \mid \beta_1(v))]dF(v)$. Working iteratively in this manner one obtains that the payoff to this punishment is:

$$\sum_{i=1}^{\infty} \delta^{i-1} \int_0^a H(v)Q(v;\beta_i)dF(v).$$

Incentive compatibility requires $Q(\cdot;\beta)$ to be non decreasing. From Lemma 4 in the appendix the above expression is minimized when we rule out constant bidding regions, i.e. when the bidding rules are constrained to induce efficiency so that $Q(v;\beta) = F(v)^{n-1}$. The single stage Nash strategy accomplishes this task while satisfying incentive compatibility. Note that when bidding rules are constrained to induce efficiency payoffs are necessarily equal to the Nash equilibrium stage game payoffs.

The next step is to show that participants always bid with probability one. Suppose a mechanism which randomly selects agents not to bid. The identity of the winning bidder is unobservable, so future gains can only be a function of the winning bid. Therefore for a certain agent with a given type to be indifferent between bidding and not bidding, all agents with lower valuation must strictly prefer not bidding and all agents with higher valuation must strictly prefer bidding. Consider a mechanism that in the first round outlaws bidding below $r > 0$, and let μ_0 be the future expected gains if nobody bids. Admissibility of the mechanism implies:

$$\delta\mu_0 F(r)^{n-1} \geq rF(r)^{n-1} + \delta\underline{U}F(r)^{n-1}.$$

And since $F(r)^{n-1} > (1 - F(s))F(s)^{n-1}$ for all s in $[0, r]$ we have that:

$$rF(r)^{n-1} + \delta\underline{U}F(r)^{n-1} > \int_0^r (1 - F(s))F(s)^{n-1}ds + \delta\underline{U}F(r)^{n-1}.$$

This previous expression represents using the Nash bidding strategy in the interval $[0, r]$. Therefore we have contradicted the supposition that outlawing bidding in a certain region dominated using the Nash bidding strategy. $\square$

Proof of Lemma 2 i) Fix $K^* > K$. Obviously any $Q(\cdot;\beta)$ satisfying constraint (5) for K will satisfy (5) for K^*.

ii) When $K = 0$ the left hand side of (5) is zero, so in order for the constraint to be respected the right hand side must be less than or equal to zero. By proposition 3, the only bidding function satisfying this requirement is the single stage Nash equilibrium bidding function.

iii) Incentive compatibility implies that due to informational asymmetries collusive gains can be no higher than those from using a scheme awarding the good to all types with equal probability (McAfee & McMillan (1992) Theorem 1). When such a scheme is used, a necessary and sufficient condition for all types to respect admissibility is for type a to respect admissibility. For type a to respect admissibility implies $K \geq a[1-(1/n)]$.

iv) Suppose Q is the function derived from the bidding function which solves (4) subject to (5). Let Q advise bidding a constant amount on $[v_0, v_1]$, $[v_2, v_3], \ldots [v_{m-1}, v_m]$. $\Psi(K)$ can be written:

$$\Psi(K) = \sum_{i=1}^{m} \int_{v_{2(i-1)}}^{v_{2i-1}} [1-F(v)][Q(v_{i-1}, v_i) - F(v)^{n-1}]dv =$$
$$\sum_{i=1}^{m} \int_{v_{2(i-1)}}^{v_{2i-1}} [F(v)^{n-1} - Q(v_{i-1}, v)] \left(\int_{v_{2(i-1)}}^{v} [1-F(x)]dx \frac{f(v)}{F(v)-F(v_{i-1})} - [1-F(v)] \right) dv. \tag{7}$$

The above equality is obtained from the fact that $\frac{\partial Q(v_{i-1}, v)}{\partial v} = [F(v)^{n-1} - Q(v, v_{i-1})] \frac{f(v)}{F(v)-F(v_{i-1})}$.

From inequality (5) we have that

$$\frac{Q(v;\beta)}{\int_0^v Q(s;\beta)ds}F(v)^{n-1}K \geq F(v)^{n-1} - Q(v_{2(i-1)}, v) \quad \forall v \in [v_{2(i-1)}, v_{2i-1}].$$

Since $Q(v_{2(i-1)}, v) \leq F(v)^{n-1}$ and $Q(s, \beta) \geq F(s)^{n-1}/n$, we have:

$$\gamma K \geq nK\frac{F(v)^{n-1}F(v)^{n-1}}{\int_0^v F(s)^{n-1}ds} \geq [F(v)^{n-1} - Q(v_{2(i-1)}, v)],$$

where $\gamma = \max_v : \frac{nF(v)^{2n-2}}{\int_0^v F(s)^{n-1}ds}$. Note that γ is finite. Substituting this into equation (7) we obtain:

$$\Psi(K) \leq \gamma K \sum_{i=1}^{m} \int_{v_{2(i-1)}}^{v_{2i-1}} \left(\int_{v_{2i-1}}^{s} \left[\frac{1-F(x)}{f(x)} - \frac{1-F(s)}{f(s)} \right] \frac{dF(x)}{F(s) - F(v_{2(i-1)})} \right) dF(s).$$

Since the term in brackets is decreasing in $v_{2(i-1)}$, we have

$$\Psi(K) \leq \gamma K \int_0^a \int_0^s \left(\frac{1-F(x)}{f(x)} - \frac{1-F(s)}{f(x)} \right) \frac{dF(x)}{F(s)} dF(s).$$

This proves the existence of a finite number, θ such that $\Psi(K) \leq \theta K$. $\square$

The proof of proposition 6 makes reference to the following lemma which we state and prove before the proof of proposition 6.

Lemma 4 *For any $0 < v_0' < v_1 < a$ it is the case that:*

$$\int_{v_0'}^{v_1} H(v)\frac{F(v_1)^n - F(v_0')^n}{n(F(v_1) - F(v_0'))}dF(v) \geq \int_{v_0'}^{v_1} H(v)F(v)^{n-1}dF(v).$$

Proof of Lemma 4 After an integration by parts the above inequality can be written:

$$\int_{v_0'}^{v_1} H'(v)\frac{F(v_1)-F(v)}{(F(v_1)-F(v_0'))}dv \geq \int_{v_0'}^{v_1} H'(v)\frac{F(v_1)^n-F(v)^n}{(F(v_1)^n-F(v_0')^n)}dv.$$

Since $H'(v) < 0$, it suffices to show:

$$\frac{F(v_1)-F(v)}{(F(v_1)-F(v_0'))} \leq \frac{F(v_1)^n-F(v)^n}{(F(v_1)^n-F(v_0')^n)} \quad \forall v \in (v_0', v_1).$$

Since it is the case that $F(v_0') < F(v) < F(v_1)$, $F(v)$ can be written:

$$F(v) = \lambda F(v_0') + (1-\lambda)F(v_1) \quad \text{for some } \lambda \in (0,1).$$

Therefore after substitution and a rearrangement it is sufficient to show that:

$$\begin{aligned}&[\lambda F(v_0') + (1-\lambda)F(v_1)]^n[F(v_1) - F(v_0')] \\ &\qquad\qquad \leq [\lambda F(v_0')^n + (1-\lambda)F(v_1)^n][F(v_1) - F(v_0')]\end{aligned}$$

Which is always satisfied since $F(v)^n$ is a convex function of $F(v)$. $\square$

Proof of Proposition 4 Suppose an optimal collusive bidding function which is continuously increasing for some interval $[v_0, v_1] \subset [0, a]$. To show a contradiction, modify the bidding rule so that it awards the object to types in $[v_0, v_1]$ with equal probability and respects incentive compatibility. There are two cases which will be considered separately: 1) Type v_1 is weakly prefers the old scheme. 2) Type v_1 strictly prefers the new scheme. In each treatment incentive compatibility is verified while using the original value $\overline{U}$, then it is shown that the new bidding function generates a higher $\overline{U}$.

Case 1 Suppose that after invoking b_0', type v_1 is just as well off as under the old bidding function. Therefore bidding levels b_1 and b_2 can be maintained and all incentive compatibility constraints are satisfied. Now apply lemma 4 to see that the expected value to collusion has been increased. Suppose that type v_1 strictly prefers the old scheme. Then if b_1 and b_2 are held constant, the type indifferent between bidding b_0' and b_1 will be located to the left of v_1. Now raise b_1 and b_2 such that the indifferent type between bidding b_0' and b_1 remains v_1 and such that the indifferent type between bidding b_1 and b_2 remains v_2. Notice that this change in the bidding function has not changed the probabilities that any type greater than v_1 wins the auction and all incentive compatibility and admissibility constraints are respected. Now apply lemma 4.

Case 2 Suppose that type v_1 strictly prefers the new scheme. Then the type indifferent between bidding b_0' and b_1 moves to the right of v_1. Call this new indifferent type v_1'. Incite v_1' to move back to v_1 by decreasing the continuation payoff if the winning bid is b_0'. Such a change produces a more profitable bidding function by lemma 4, but may not produce higher rents. Invoke Proposition 4 to be assured of the existence of a collusive mechanism with a bidding function which is never continuously increasing and having the bang-bang property. This bang-bang mechanism uses a bidding function which is even more profitable. Furthermore, such a bang-bang mechanism generates $\overline{U} = \int_0^a H(v)Q(v;\beta^*)dF(v)$, where β^* is the newest bidding function. So since the bang-bang mechanism produces rents which are generated by a bidding function which dominates the original scheme, the proposition is proved. $\square$

Proof of Lemma 3 Suppose there does not exist an indifferent type under an optimal collusive bidding rule. Call this rule β and denote the single stage profits it generates by $\int_0^a H(v)Q(v;\beta)dF(v)$. Due to the lack of an indifferent type, there exists $\eta > 0$ such that β' is incentive compatible if it satisfies $\int_0^a |\beta(v) - \beta'(v)| dv < \eta$. Now we can choose a (finite) sequence of bidding functions $(\beta^k)_{k=1}^K$ such that for all $k \in \{1, 2, \ldots, K-1\}$: 1) $\int_0^a |\beta^k(v) - \beta^{k+1}(v)| dv < \eta$, 2) $\beta^K(v) = 0$ for all v, 3) $\beta^1 = \beta$, and 4) $\int_0^a H(v)(Q(v; \beta^{k+1}) - Q(v; \beta^k)) dF(v) > 0$. We are assured of 4) because $H(v)$ was assumed strictly decreasing, and we are assured that $K > 1$ because there is at least one discontinuity in the bidding function. Obviously this sequence contains at least two bidding functions which are incentive compatible and at least one which gives higher single stage payoffs than β, contradicting the hypothesis on the optimality of β. $\square$

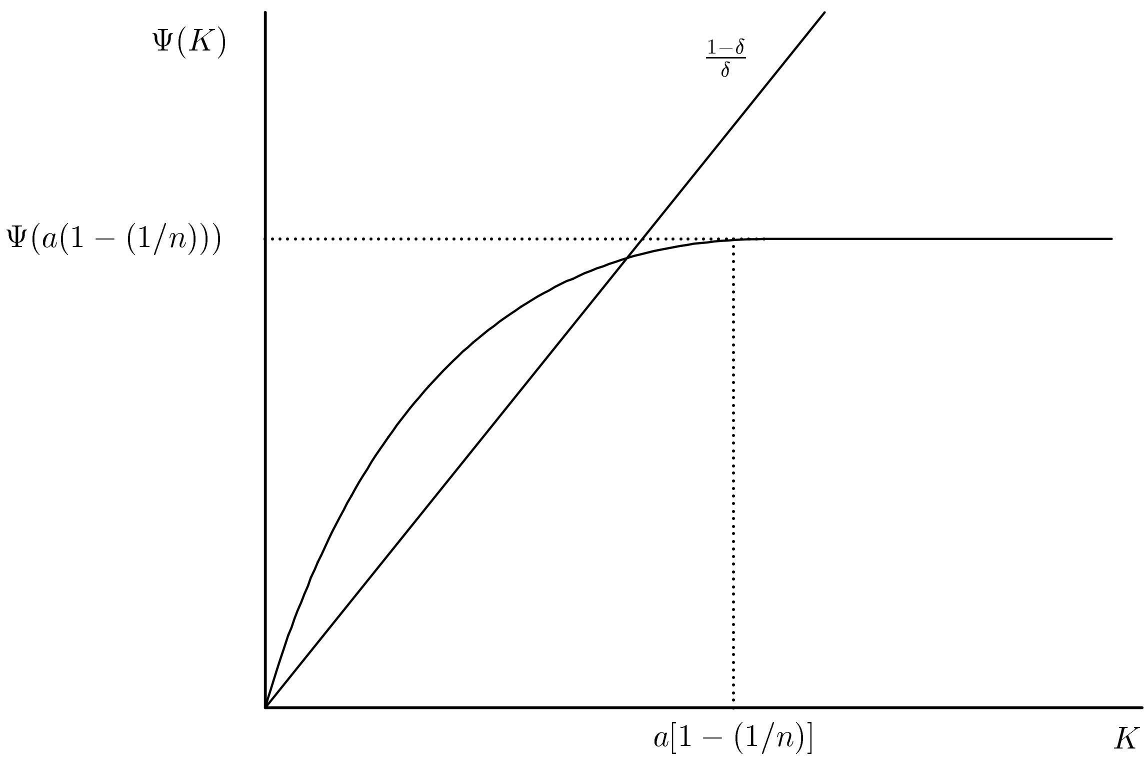

Figure 1: Graph of $\Psi(K)$

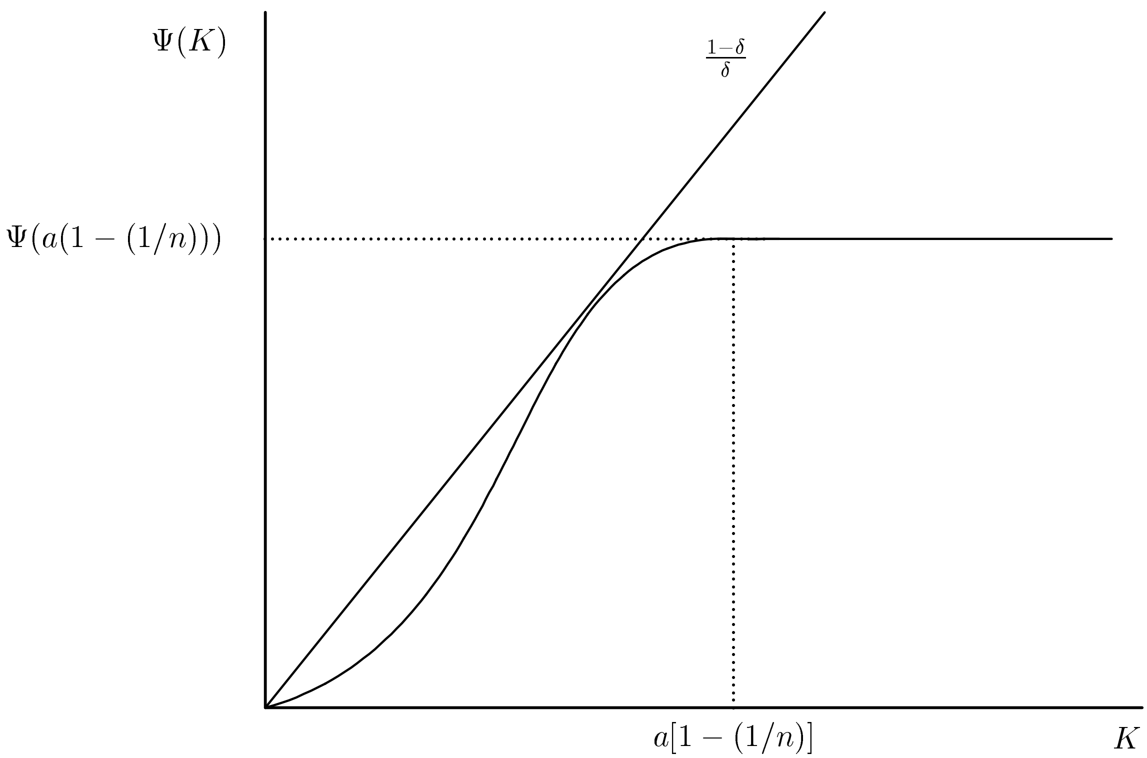

Figure 2: Graph of $\Psi(K)$

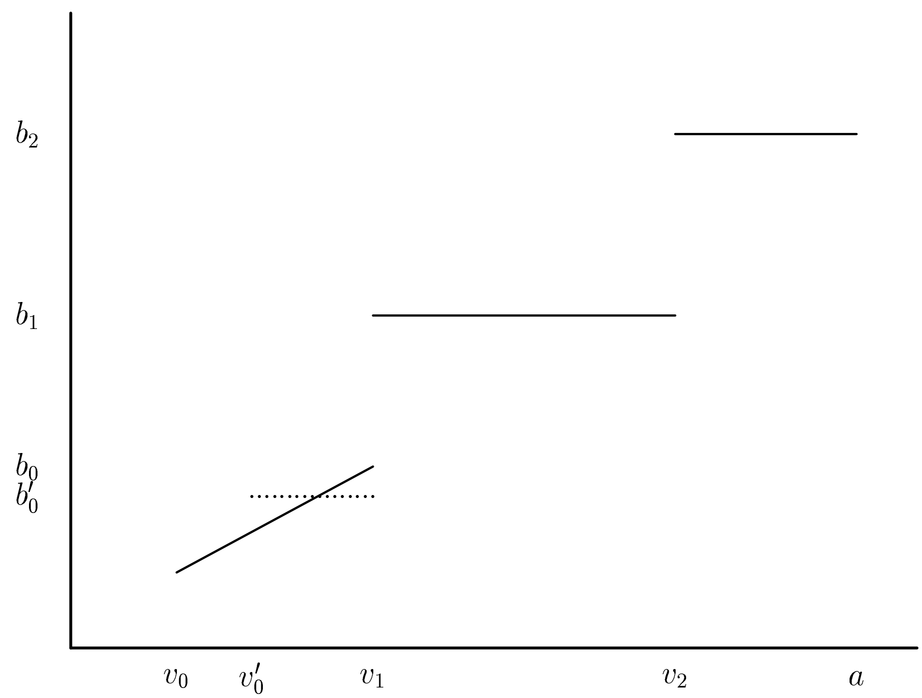

Figure 3: A bidding function

Notes

[1]In fact it is difficult to give an example of a purely one shot auction.

[2]Under usual hypotheses identical bids should be observed with zero probability.

[3]In a recent paper Athey et al. (1998) have explored this issue.

[4]This is the case for most common distributions. See Bagnoli & Bergstrom (1989) for more details.

[5]Quoting from Wilson (1992): "...for symmetric first-price auctions there is some presumption that the symmetric equilibrium is the unique equilibrium."

[6]Note that contrary to Abreu et al. (1986) the bang-bang property is defined so that it invokes the harshest punishment as well. This is not a necessary condition for Proposition 4, but a sufficient one. This is because a less than maximal punishment may suffice to ensure obedience to a bidding rule specifying bidding zero regardless of valuation. To see that a maximal punishment is sufficient, notice that deviation will be observed with probability zero thus will not affect payoffs. Furthermore, a harsher punishment can always support a scheme supportable by a weaker punishment. Note that if the admissibility constraint is binding, then the harshest possible punishments must be used as well.

[7]See the discussion in section II. of Graham & Marshall (1987).

[8]It can be shown that profits move continuously in the bidding ceiling and reserve price. Since we can consider a compact region over which to select the reserve price, we are assured of the existence of a maximum.

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