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RÉSUMÉ

Dans des environnements spatiaux, nous considérons des fonctions de bien-être social satisfaisant les hypothèses d'Arrow, i.e. *la faiblesse au sens de Pareto et l'indépendance des alternatives non pertinentes*. Lorsque l'espace des politiques est un continuum unidimensionnel, une telle fonction de bien-être est déterminée par une collection de 2^N préférences strictement quasi-concaves et une règle de bris d'égalité. Comme corollaire, nous obtenons que, lorsque le nombre d'électeurs est impair, le vote à la majorité simple est transitif si et seulement si la préférence de chaque électeur est strictement quasi concave. Lorsque l'espace des politiques est multidimensionnel, nous établissons le théorème d'impossible d'Arrow. Nous montrons, entre autres, que *la faiblesse au sens de Pareto, l'indépendance des alternatives non pertinentes* et *la non-dictature* sont incompatibles si l'ensemble des alternatives possède un intérieur non vide et est compact et convexe.

Mots clés : théorème d'Arrow, indépendance des alternatives non pertinentes

ABSTRACT

In spatial environments, we consider social welfare functions satisfying Arrow's requirements, i.e., *weak Pareto and independence of irrelevant alternatives*. When the policy space is a one-dimensional continuum, such a welfare function is determined by a collection of 2^N strictly quasi-concave preferences and a tie-breaking rule. As a corollary, we obtain that when the number of voters is odd, simple majority voting is transitive if and only if each voter's preference is strictly quasi-concave. When the policy space is multi-dimensional, we establish Arrow's impossibility theorem. Among others, we show that *weak Pareto, independence of irrelevant alternatives*, and *non-dictatorship* are inconsistent if the set of alternatives has a non-empty interior and it is compact and convex.

Key words : Arrow's theorem, independence of irrelevant alternatives

1 Introduction

A social welfare function is a procedure for aggregating profiles of individual preferences into social orderings. Arrow's theorem shows that it is impossible for a social welfare function to satisfy *weak Pareto* (if all individuals strictly prefer one alternative to another, then so does society), *independence of irrelevant alternatives* (the social ranking of two alternatives only depends on the individual rankings of these alternatives), and *non-dictatorship* when the preference domain is unrestricted. When the set of alternatives is structured, the assumption of unrestricted domain might be unreasonable. One important exception of this kind are spatial environments. In spatial environments, alternatives are points in a Euclidean space of issue positions and individual preferences are continuous, quasi-concave, and have bliss points. Although the assumption of unrestricted domain is unreasonable in economic and spatial environments, it has been shown that Arrow's theorem remains valid in most of these environments.[1]

There is one well-known spatial environment in which Arrow's requirements are consistent. If the number of voters is odd and the policy space is one-dimensional, then simple majority voting is transitive and satisfies *weak Pareto*, *independence of irrelevant alternatives*, and *anonymity* (Black ,1948; Arrow, 1951, 1963). Simple majority voting is an example of an Arrovian welfare function belonging to the following class: if the number of voters is n, fix $n-1$ preferences of constant voters, and apply to each profile of individual preferences majority voting over this profile and the $n-1$ fixed voters. However, a characterization of all welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives* was missing. Our paper fills this gap. The class of welfare functions described above, in which the preferences of the fixed voters are strictly quasi-concave, is characterized by *weak Pareto*, *independence of irrelevant alternatives*, and *anonymity* up to a tie-breaking rule. If *anonymity* is dropped, then instead of $n-1$ fixed preferences we need 2^N fixed strictly quasi-concave

[1] An excellent review of the literature is Le Breton and Weymark (2000a).

preferences.

A key step in the proof of Arrow's theorem is that decisiveness of a coalition spreads from one pair of alternatives to all pairs of alternatives. In proving the previous fact we need to assume that each individual's preference domain is unrestricted. This is the difference between one-dimensional spatial environments and environments with an unrestricted domain. We show that in one-dimensional spatial environments decisiveness of a coalition spreads in the following way: if a coalition is decisive over "a preferred to b" (where $a, b \in \mathbb{R}$ are such that $a < b$), then it is decisive over "a preferred to c" for all $c > b$ and over "c preferred to b" for all $a < c < b$. This implies that for each coalition S, there is a point $x_S \in [-\infty, +\infty]$ such that S is decisive over "a preferred to b" for all $x_S < a < b$ and $N \backslash S$ is decisive over "c preferred to d" for all $d < c < x_S$. Therefore, when the peaks of S converge to $-\infty$ and the peaks of $N \backslash S$ to $+\infty$, then an Arrovian welfare function chooses a strictly quasi-concave ordering with quasi bliss point x_S. The collection of 2^N strictly quasi-concave preferences characterizes an Arrovian welfare function up to some tie-breaking rule. Here, the key is to show that if the ranking over two alternatives is not determined, then this does not cause any intransitivities, no matter what ranking we choose between these two alternatives. Of course, this is only possible if the two alternatives belong to opposite sides of the quasi bliss point of the social ordering and could form an indifference class.

Our characterization is one of the few positive results in Arrovian social choice. A corollary of our result is that simple majority voting is transitive if and only if individual preferences are strictly quasi-concave.

If the policy space is multi-dimensional and unbounded, and preferences are Euclidean, then Arrow's requirements are inconsistent (Border, 1984). Other proofs show that the domain of Euclidean preferences is "saturating" and apply a result due to Kalai, Muller, and Satterthwaite (1979) to deduce Arrow's impossibility (Le

Breton and Weymark, 1996, 2000a,b; Campbell, 1993[2]). However, the proofs of these results rely importantly on the unboundedness of the policy space. As far as we know, Arrow's theorem has not been established in spatial environments when the set of alternatives is compact and convex. It is natural to assume that a government faces budget constraints and therefore the set of policy alternatives becomes bounded. We derive Arrow's theorem in such spatial environments. If individual preferences are Euclidean and the set of alternatives has a non-empty interior and it is compact and convex, then *weak Pareto*, *independence of irrelevant alternatives*, and *non-dictatorship* are inconsistent, i.e. again decisiveness of a coalition spreads from one pair of alternatives to all pairs of alternatives and dictatorship results.

The paper is organized as follows. In Section 2 we introduce our notation and the main definitions. In Section 3 we characterize the welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives* if the policy space is one-dimensional. In Section 4 we show that *weak Pareto*, *independence of irrelevant alternatives*, and *non-dictatorship* are inconsistent if the policy space is compact, convex, and at least two-dimensional. Section 5 concludes.

2 Notation and Definitions

We use the same terminology and notation as Le Breton and Weymark (2000a). Let $N \equiv \{1, 2, \ldots, n\}$ denote a finite set of agents with $n \geq 2$, and let $A \subseteq \mathbb{R}^m$ denote a set of alternatives. Each point in $\mathbb{R}^m$ identifies the changes in the level of m different policies, for example public spending on police, health care, and so on. Let $\mathcal{W}$ denote the set of all complete and transitive relations over A. An element of $\mathcal{W}$ is called a *weak ordering* over A. Given $R_i \in \mathcal{W}$, the corresponding strict relation, P_i, and the indifference relation, I_i, are defined as follows: for all $a, b \in A$, (i) $aP_ib \Leftrightarrow \neg bR_ia$, and (ii) $aI_ib \Leftrightarrow aR_ib$ and bR_ia. Note that if $\neg aR_ib$, then by completeness of R_i, aR_ib. Hence, $\neg bR_ia$ is enough to describe the strict relation P_i. Let $\mathcal{R} \subseteq \mathcal{W}$.

[2]Campbell (1993) drops *weak Pareto* and imposes instead continuity of social preference.

A *(preference) profile* is a list $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$. A *(social) welfare function* associates with each profile a weak ordering over A. Formally, a welfare function is a mapping $f : \mathcal{R}^N \to \mathcal{W}$ such that for all $R \in \mathcal{R}^N$, $f(R) \in \mathcal{W}$. We call $f(R)$ the *social ordering* (that f associates with R). Note that $f(R)$ need not belong to the individual preference domain $\mathcal{R}$. Other authors impose the restriction that any social ordering belongs to each individual's preference domain (for example, Peters, van der Stel, and Storcken, 1992, and Bossert and Weymark, 1993).

Arrow's requirements are as follows. The first axiom says that if all agents strictly prefer a to b, then a should also be socially strictly preferred to b.

Weak Pareto: For all $R \in \mathcal{R}^N$ and all $a, b \in A$, if for all $i \in N$, aP_ib, then $\neg bf(R)a$.

Given $R \in \mathcal{R}^N$, $X \subseteq A$, and $j \in N$, let $R_j|X$ denote the restriction of R_j to X, and $R|X \equiv (R_i|X)_{i \in N}$. The second axiom says that the social ordering of two alternatives only depends on the profile of individual preferences restricted to these two alternatives.

Independence of Irrelevant Alternatives: For all $R, \bar{R} \in \mathcal{R}^N$ and all $a, b \in A$, if $R|\{a,b\} = \bar{R}|\{a,b\}$, then $f(R)|\{a,b\} = f(\bar{R})|\{a,b\}$.

A welfare function is *Arrovian* if it satisfies *weak Pareto* and *independence of irrelevant alternatives*. A welfare function is *dictatorial* if there exists some agent such that for each profile the social strict preference relation respects the strict preference relation of this agent.

Non-Dictatorship: There exists no $i \in N$ such that for all $R \in \mathcal{R}^N$ and all $a, b \in A$, if aP_ib, then $\neg bf(R)a$.

A welfare function treats individuals symmetrically if for all permutations of individuals' preferences, the social ordering remains unchanged.

Anonymity: For all $R \in \mathcal{R}^N$ and all permutations σ of N, $f(\sigma(R)) = f(R)$.

3 One-Dimensional Policy Spaces

In this section the policy space is a one-dimensional continuum. Specifically, the set of alternatives is the set of real numbers $\mathbb{R}$. For example, in an election each element of $\mathbb{R}$ represents a candidate's political ideology on a left-right spectrum.

Given $i \in N$, a weak ordering $R_i \in \mathcal{W}$ is *single-peaked* if there exists a point $p(R_i) \in \mathbb{R}$, called the peak of R_i, such that for all $a, b \in \mathbb{R}$, if $a < b \leq p(R_i)$ or $p(R_i) \leq b < a$, then bP_ia. Let $\mathcal{R}$ denote the set of all single-peaked preferences over $\mathbb{R}$. Each agent $i \in N$ has a single-peaked preference relation over $\mathbb{R}$. A single-peaked preference $R_i \in \mathcal{R}$ is *symmetric* if for all $a, b \in \mathbb{R}$, $aR_ib \Leftrightarrow |a - p(R_i)| \leq |b - p(R_i)|$.

We will give a complete characterization of all welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives*. In this characterization the following weak orderings are important.

A weak ordering $R_i \in \mathcal{W}$ is *strictly quasi-concave*, if for all $a, b, c \in \mathbb{R}$ such that $a < b < c$, we have $\neg aR_ib$ or $\neg cR_ib$. In other words, b is never a worst alternative in $\{a, b, c\}$ and the restriction of R_i to $\{a, b, c\}$ is single-peaked . Let $\mathcal{C}$ denote the set of all strictly quasi-concave orderings over $\mathbb{R}$.

Lemma 3.1 *Let $R_i \in \mathcal{W}$. The weak ordering R_i is strictly quasi-concave if and only if there exists a quasi bliss point $p(R_i) \in \mathbb{R} \cup \{-\infty, +\infty\}$ (for convenience we use the same notation as for peaks) such that (i) for all $a, b \in \mathbb{R}$, if $a < b < p(R_i)$ or $p(R_i) < b < a$, then bP_ia, and (ii) if $p(R_i) \in \mathbb{R}$, then either (for all $x \in]-\infty, p(R_i)[$, $p(R_i)P_ix$), or (for all $x \in]p(R_i), +\infty[$, $p(R_i)P_ix$).*

Proof. It is easy to check that R_i is strictly quasi-concave if (i) and (ii) hold.

Let R_i be strictly quasi-concave. For all $x \in \mathbb{R}$, let $B(x, R_i) \equiv \{y \in \mathbb{R} \,|\, xR_iy\}$. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that for all $k \in \mathbb{N}$, if $B(x_k, R_i) \neq \mathbb{R}$, then $B(x_k, R_i) \subsetneq B(x_{k+1}, R_i)$, and for all $y \in \mathbb{R}$, there exists $k \in \mathbb{N}$ such that $y \in B(x_k, R_i)$. Without loss of generality, let $(x_k)_{k\in\mathbb{N}}$ converge to $p(R_i)$ (otherwise we choose a convergent subsequence of $(x_k)_{k\in\mathbb{N}}$). We show (i) and (ii).

Let $a, b \in \mathbb{R}$ be such that $a < b < p(R_i)$ or $p(R_i) < b < a$. Then there exists $k \in \mathbb{N}$ such that $b \in B(x_k, R_i)$ and $|x_k - p(R_i)| < |b - p(R_i)|$. Thus, x_kR_ib and $a < b < x_k$ or $x_k < b < a$. Because R_i is strictly quasi-concave, bP_ia, which is (i).

Suppose that $p(R_i) \in \mathbb{R}$ but (ii) does not hold. Then for some $a, b \in \mathbb{R}$ we have $a < p(R_i) < b$, $aR_ip(R_i)$, and $bR_ip(R_i)$. Obviously, this is in contradiction to strict quasi-concavity of R_i. □

Obviously, $\mathcal{R} \subseteq \mathcal{C}$. However, a strictly quasi-concave preference R_i may not be single-peaked, even if $p(R_i) \in \mathbb{R}$. For example, let $u : \mathbb{R} \to \mathbb{R}$ be such that for all $x \in \mathbb{R}$, $u(x) \equiv \frac{1}{x}$ if $x > 0$ and $u(x) = x$ if $x \leq 0$. Let $R_u \in \mathcal{W}$ be such that for all $a, b \in \mathbb{R}$, $aR_ub \Leftrightarrow u(a) \geq u(b)$. Then $R_u \in \mathcal{C}\backslash\mathcal{R}$.

3.1 Decisiveness

Given $S \subseteq N$ and $a, b \in A$, we say that S is *semi-decisive* over "a preferred to b" if there exists $R \in \mathcal{R}^N$ such that (i) for all $i \in S$, aP_ib, (ii) for all $i \in N\backslash S$, bP_ia, and (iii) $af(R)b$.[3] Let $D^s(a, b)$ denote the set of all coalitions that are semi-decisive over "a preferred to b". Given $a, b \in A$ and $R \in \mathcal{R}$, if $\neg bf(R)a$ whenever aP_ib for all $i \in S$, then S is said to be *decisive* over "a preferred to b". Let $D(a, b)$ denote the set of all coalitions that are decisive over "a preferred to b".

One of the important steps in Arrow's impossibility theorem is that whenever a coalition $S \subseteq N$ is decisive over "a preferred to b", then it is also decisive over "a

[3]The literature on Arrovian social choice often refers to our definition as "a coalition is semi-decisive over the pair (a, b)".

preferred to c" and "c preferred to b" for any other alternative c. Therefore, every coalition is either decisive over every pair of alternatives or over no pair.

Because here individual preferences are restricted to be single-peaked and the policy space is one-dimensional, decisiveness of a coalition does not spread over all pairs of alternatives as in Arrow's original theorem. Decisiveness of a coalition expands in a weaker form. Given $a, b \in \mathbb{R}$ such that $a < b$, if S is semi-decisive over "a preferred to b", then S is decisive over "a preferred to c" for all $c \in]b, +\infty[$ and S is decisive over "c preferred to b" for all $c \in]a, b[$.

Lemma 3.2 *Let $f : \mathcal{R}^N \to \mathcal{W}$ be a welfare function satisfying weak Pareto and independence of irrelevant alternatives. Let $S \subseteq N$ and $a, b, c \in \mathbb{R}$ be such that $a < b < c$ or $a > b > c$.*

(i) If $S \in D^s(a, b)$, then $S \in D(a, c)$.

(ii) If $S \in D^s(a, c)$, then $S \in D(b, c)$.

Proof. First, we show (i). Let $R \in \mathcal{R}^N$ be such that for all $i \in S$, aP_ic. We have to show that $\neg cf(R)a$. Let $\tilde{R} \in \mathcal{R}^N$ be such that

(a) for all $i \in N$, $\tilde{R}_i|\{a, c\} = R_i|\{a, c\}$;

(b) for all $i \in S$, $a\tilde{P}_ib$ and $b\tilde{P}_ic$; and

(c) for all $i \in N \backslash S$, $b\tilde{P}_ia$ and $b\tilde{P}_ic$.

It is easy to check that we can find $\tilde{R} \in \mathcal{R}^N$ such that (a) to (c) are satisfied (for all $i \in N$, we can even choose $\tilde{R}_i$ to be symmetric). By (b) and (c), for all $i \in N$, $b\tilde{P}_ic$. Thus, by *weak Pareto*, $\neg cf(\tilde{R})b$. Because $S \in D^s(a, b)$ and f satisfies *independence of irrelevant alternatives*, (b) and (c) imply $af(\tilde{R})b$. Thus, by transitivity of $f(\tilde{R})$, $\neg cf(\tilde{R})a$. Hence, by (a) and *independence of irrelevant alternatives*, $\neg cf(R)a$ and $S \in D(a, c)$, the desired conclusion.

Second, we show (ii). Let $R \in \mathcal{R}^N$ be such that for all $i \in S$, bP_ic. We have to show that $\neg cf(R)b$. Let $\tilde{R} \in \mathcal{R}^N$ be such that

(a) for all $i \in N$, $\tilde{R}_i|\{b,c\} = R_i|\{b,c\}$,

(b) for all $i \in S$, $b\tilde{P}_i a$ and $a\tilde{P}_i c$, and

(c) for all $i \in N\backslash S$, $b\tilde{P}_i a$ and $c\tilde{P}_i a$.

(Again, for all $i \in N$, $\tilde{R}_i$ can be chosen to be symmetric.) By *weak Pareto*, $\neg a f(\tilde{R}) b$. Because $S \in D^s(a,c)$, we have $a f(\tilde{R}) c$. Thus, by transitivity of $f(\tilde{R})$, $\neg c f(\tilde{R}) b$. By (a) and *independence of irrelevant alternatives*, $\neg c f(R) b$. Hence, $S \in D(b,c)$, the desired conclusion. □

3.2 Social Orderings at Maximal Conflicts

In avoiding limits of profiles we add two non-single-peaked preferences to the set $\mathcal{R}$. Obviously, $\geq$ and $\leq$ are preferences over $\mathbb{R}$, where $\geq$ has a bliss point at $+\infty$ and $\leq$ at $-\infty$.

Let $\tilde{\mathcal{R}} \equiv \mathcal{R} \cup \{\geq, \leq\}$. The following lemma shows that the addition of $\geq$ and $\leq$ to $\mathcal{R}$ has no influence on a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*.

Lemma 3.3 *Let $f : \mathcal{R}^N \to \mathcal{W}$ be a welfare function satisfying* weak Pareto *and* independence of irrelevant alternatives. *Define $\tilde{f} : \tilde{\mathcal{R}}^N \to \mathcal{W}$ as follows: for all $R \in \tilde{\mathcal{R}}^N$ and all $a, b \in \mathbb{R}$, take $\bar{R} \in \mathcal{R}^N$ such that $\bar{R}|\{a,b\} = R|\{a,b\}$ and set*

$$\tilde{f}(R)|\{a,b\} \equiv f(\bar{R})|\{a,b\}.$$

Then $\tilde{f}$ is a well-defined welfare function satisfying weak Pareto *and* independence of irrelevant alternatives. *Moreover, $\tilde{f}|\mathcal{R}^N = f$.*

Proof. Let $R \in \tilde{\mathcal{R}}^N$. Because f satisfies *independence of irrelevant alternatives*, for all $a, b \in \mathbb{R}$, $\tilde{f}(R)|\{a,b\}$ is well-defined. Thus, $\tilde{f}$ satisfies *weak Pareto* and *independence of irrelevant alternatives*. Obviously, $\tilde{f}(R)$ is complete. It remains to show that $\tilde{f}(R)$ is transitive. Let $a, b, c \in \mathbb{R}$ be such that $a \leq b \leq c$. Let $\bar{R} \in \mathcal{R}^N$ be such that

for all $i \in N$, (i) if $R_i = \leq$, then $p(\bar{R}_i) = a$, (ii) if $R_i \in \mathcal{R}$, then $\bar{R}_i = R_i$, and (iii) if $R_i = \geq$, then $p(\bar{R}_i) = c$. By definition, $\tilde{f}(R)|\{a,b,c\} = f(\bar{R})|\{a,b,c\}$ and $\tilde{f}(R)$ is transitive. □

Given a coalition $S \subseteq N$, we consider the profile where the members belonging to S announce $\leq$ and the other agents announce $\geq$. Clearly, the coalitions S and $N\backslash S$ disagree on every pair of different alternatives. We show that the social orderings at these profiles of maximal conflicts are strictly quasi-concave. They also satisfy a certain monotonicity property and determine for each profile the social ordering up to some tie-breaking when social preference can be chosen to be arbitrary.

Throughout the remaining part of Subsection 3.2, let f be a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*. For all $S \subseteq N$, let $(\leq_S, \geq_{N\backslash S})$ be the profile $R \in \tilde{\mathcal{R}}^N$ such that for all $i \in S$, $R_i = \leq$, and for all $i \in N\backslash S$, $R_i = \geq$. Then for all $S \subseteq N$, let $Q^S \equiv \tilde{f}(\leq_S, \geq_{N\backslash S})$. Let $Q \equiv \{Q^S \mid S \subseteq N\}$. We call Q a *collection of calibration relations*. We will show that Q completely determines f (up to some tie-breaking). By Lemma 3.3, $\tilde{f}$ satisfies *weak Pareto*. Thus, $Q^{\emptyset} = \geq$ and $Q^N = \leq$. The following lemma specifies some properties of the collection Q.

Lemma 3.4 *(i) The collection Q is (inclusive) monotone, i.e. for all $S, T \subseteq N$ such that $S \subseteq T$, and all $a, b, c, d \in \mathbb{R}$ such that $a < b < c < d$, (i.i) if $bQ^S c$, then $\neg dQ^T b$; (i.ii) if $bQ^S d$, then $\neg dQ^T c$; (i.iii) if $cQ^T b$, then $\neg aQ^S c$; and (i.iv) if $cQ^T a$, then $\neg aQ^S b$.*

(ii) For all $S \subseteq N$, $Q^S \in \mathcal{C}$.

(iii) For all $S, T \subseteq N$, if $S \subseteq T$, then $p(Q^T) \leq p(Q^S)$.

Proof. (i) By definition, $Q^S = \tilde{f}(\leq_S, \geq_{N\backslash S})$ and $Q^T = \tilde{f}(\leq_T, \geq_{N\backslash T})$. If $bQ^S c$, then $S \in D^s(b,c)$. Thus, by Lemma 3.2, $S \in D(b,d)$. Hence, by $S \subseteq T$, we have $\neg dQ^T b$. Statements (i.ii), (i.iii), and (i.iv) follow similarly.

(ii) Let $a, b, c \in \mathbb{R}$ be such that $a < b < c$. Suppose that $aQ^S b$ and $cQ^S b$. Then by (i.i), $\neg cQ^S a$, and by (i.iii), $\neg aQ^S c$, which contradicts completeness of Q^S.

(iii) Suppose that $p(Q^T) > p(Q^S)$. Let $a, b, c \in]p(Q^S), p(Q^T)[$ be such that $a < b < c$. Then $aQ^S b$ and $S \in D^s(a, b)$. Thus, by Lemma 3.2, $S \in D(a, c)$. By $S \subseteq T$, $aQ^T c$. This is a contradiction to $a < c < p(Q^T)$ and $Q^T \in \mathcal{C}$. □

Recall that a strictly quasi-concave preference $R_i \in \mathcal{C}$ is either equal to $\leq$ or $\geq$, or there exists a *quasi bliss point* $p(R_i) \in \mathbb{R}$ such that (i) for all $a, b \in \mathbb{R}$, if $a < b < p(R_i)$ or $p(R_i) < b < a$, then $bP_i a$, and (ii) either (for all $x \in]-\infty, p(R_i)[$, $p(R_i)P_i x$), or (for all $x \in]p(R_i), +\infty[$, $p(R_i)P_i x$). We use the convention that $p(\leq) \equiv -\infty$ and $p(\geq) \equiv +\infty$.

Next, we show that for each profile the social ordering is strictly quasi-concave.

Let $R \in \mathcal{R}^N$ be such that $p(R_{i_1}) \leq p(R_{i_2}) \leq \cdots \leq p(R_{i_n})$. Let $S_0 \equiv \emptyset$ and for all $t \in \{1, 2, \ldots, n\}$, $S_t \equiv \{i_1, i_2, \ldots, i_t\}$. Thus, $S_n = N$. The *median* of $M(R) = \{p(R_{i_1}), p(R_{i_2}), \ldots, p(R_{i_n}), p(Q^{S_0}), p(Q^{S_1}), \ldots, p(Q^{S_n})\}$ is the number $m(R) \in \mathbb{R}$ such that (i) $m(R) \in M(R)$, (ii) at least $n+1$ elements of $M(R)$ are smaller than or equal to $m(R)$, and (iii) at least $n+1$ elements of $M(R)$ are greater than or equal to $m(R)$.

Theorem 3.1 *For all $R \in \mathcal{R}^N$, $f(R)$ is strictly quasi-concave with quasi bliss point $m(R)$. Furthermore, if $m(R) \notin \{p(Q^S) \mid S \subseteq N\}$, then $f(R)$ is single-peaked.*

Proof. Let $R \in \mathcal{R}^N$. Without loss of generality, suppose $p(R_1) \leq p(R_2) \leq \cdots \leq p(R_n)$. By *independence of irrelevant alternatives*, we have (i) $f(R)|]-\infty, p(R_1)] = Q^{\emptyset}|]-\infty, p(R_1)] = \geq |]-\infty, p(R_1)]$, (ii) for all $t \in \{1, 2, \ldots n-1\}$,

$$f(R)|[p(R_t), p(R_{t+1})] = Q^{S_t}|[p(R_t), p(R_{t+1})], \tag{1}$$

and (iii) $f(R)|[p(R_n), +\infty[= Q^N|[p(R_n), +\infty[= \leq |[p(R_n), +\infty[$.

Because $p(Q^{\emptyset}) = +\infty$ and $p(Q^N) = -\infty$, we have $m(R) \in [p(R_1), p(R_n)]$. Let $a, b \in \mathbb{R}$ be such that $m(R) < a < b$. We show that $\neg bf(R)a$. If $a \geq p(R_n)$, then the

assertion follows from $f(R)|[p(R_n), +\infty[= \leq |[p(R_n), +\infty[$. Let $t' \in \{1, 2, \ldots, n-1\}$ be such that $a \in [p(R_{t'}), p(R_{t'+1})[$ and $c \in]a, p(R_{t'+1})[\cap]a, b[$. We claim that $p(Q^{S_{t'}}) \leq m(R)$. If $p(Q^{S_{t'}}) > m(R)$, then by Lemma 3.4 at least $n+1$ numbers are greater than $m(R)$, namely $\{p(Q^{S_0}), p(Q^{S_1}), \ldots, p(Q^{S_{t'}}), p(R_{t'+1}), p(R_{t'+2}), \ldots, p(R_n)\}$, which contradicts the median property of $m(R)$. Thus, $p(Q^{S_{t'}}) \leq m(R) < a$. By (1), $f(R)|[p(R_{t'}), p(R_{t'+1})] = Q^{S_{t'}}|[p(R_{t'}), p(R_{t'+1})]$. Hence, $\neg c f(R) a$. Thus, $S_{t'} \in D^s(a, c)$. By Lemma 3.2 and $a < c < b$, $S_{t'} \in D(a, b)$. Hence, $\neg b f(R) a$, the desired conclusion. Similarly it can be shown that for all $a, b \in \mathbb{R}$, if $b < a < m(R)$, then $\neg b f(R) a$.

Let $t'' \in \{1, 2, \ldots, n-1\}$ be such that $m(R) \in [p(R_{t''}), p(R_{t''+1})]$. If $m(R) \in]p(R_{t''}), p(R_{t''+1})[$, then $f(R)|]p(R_{t''}), p(R_{t''+1})[= Q^{S_{t''}}|]p(R_{t''}), p(R_{t''+1})[$ implies $p(f(R)) = m(R) = p(Q^{S_{t''}})$ and $f(R) \in \mathcal{C}$. If $m(R) \in \{p(R_1), p(R_n)\}$, then (i) or (iii) implies $f(R) \in \mathcal{C}$.

Suppose that $m(R) = p(R_{t''})$ and $p(R_{t''}) \notin \{p(R_1), p(R_n)\}$. Let $S \equiv \{i \in N \,|\, p(R_i) < m(R)\}$ and $T \equiv \{i \in N \,|\, p(R_i) \leq m(R)\}$. Then $S \neq \emptyset$, $S \subsetneq T$, and $T \neq N$. Thus, $]p(R_{|S|}), p(R_{|S|+1})[\neq \emptyset$ and $]p(R_{|T|}), p(R_{|T|+1})[\neq \emptyset$. Similar arguments as above imply that $p(Q^T) \leq m(R) \leq p(Q^S)$. If $p(Q^T) < m(R)$, then by (1), for all $x \in]p(R_{|T|}), p(R_{|T|+1})[$, $\neg x f(R) m(R)$, and $f(R) \in \mathcal{C}$. If $p(Q^S) > m(R)$, then by (1), for all $x \in]p(R_{|S|}), p(R_{|S|+1})[$, $\neg x f(R) m(R)$, and $f(R) \in \mathcal{C}$. Suppose that $p(Q^T) = m(R) = p(Q^S)$. By (1), $f(R)|]p(R_{|S|}), p(R_{|S|+1})] = Q^S|]p(R_{|S|}), p(R_{|S|+1})]$ and $f(R)|[p(R_{|T|}), p(R_{|T|+1})[= Q^T|[p(R_{|T|}), p(R_{|T|+1})[$. If $f(R) \notin \mathcal{C}$, then for some $a \in]p(R_{|S|}), p(R_{|S|+1})[$, $a f(R) m(R)$, and for some $b \in]p(R_{|T|}), p(R_{|T|+1})[$, $b f(R) m(R)$. Thus, $S \in D^s(a, m(R))$ and $N \backslash T \in D^s(b, m(R))$. Hence, by Lemma 3.2, $S \in D(a, b)$ and $N \backslash T \in D(a, b)$. Therefore, $\neg a f(R) b$ and $\neg b f(R) a$, which contradicts completeness of $f(R)$. Hence, $f(R) \in \mathcal{C}$.

If $m(R) \notin \{p(Q^{S'}) \,|\, S' \subseteq N\}$, then $p(Q^T) < m(R) < p(Q^S)$. Hence, by (1), $f(R) \in \mathcal{R}$. □

Remark 3.1 If for all $S \subseteq N$, $Q^S \in \{\leq, \geq\}$, then for all $R \in \mathcal{R}^N$, $f(R)$ is a single-

peaked preference relation. $\triangleleft$

By Theorem 3.1, for all $R \in \mathcal{R}^N$, the collection Q determines the social ordering $f(R)$ restricted to $]-\infty, m(R)[$ and to $]m(R), +\infty[$ (and either on $]-\infty, m(R)]$ or $[m(R), +\infty[$). However, the quasi bliss point $m(R)$ needs not be the socially most preferred alternative. Next we show that Q also determines the ranking of $f(R)$ over all pairs (up to some arbitrary tie-breaking).

Lemma 3.5 *Let $R \in \mathcal{R}^N$ and $a, b \in \mathbb{R}$ be such that $a < b$. Let $S \equiv \{i \in N \mid aP_ib\}$, $U \equiv \{i \in N \mid bP_ia\}$, and $T \equiv N \backslash (S \cup U)$. Then the following holds.*

(i) If for some $x \in]a, b[$, aQ^Sx, or for some $x \in]-\infty, a[$, xQ^Sb, then $\neg bf(R)a$.

(ii) If for some $x \in]a, b[$, $bQ^{S\cup T}x$, or for some $x \in]b, +\infty[$, $xQ^{S\cup T}a$, then $\neg af(R)b$.

(iii) If the presumptions of (i) and (ii) do not hold, then for all $x \in]a, b[$, we have $\neg af(R)x$ and $\neg bf(R)x$, and for all $x \in]-\infty, a[\cup]b, +\infty[$, we have $\neg xf(R)a$ and $\neg xf(R)b$.

Proof. By definition, for all $i \in T$, aI_ib.

First, we show (i). If for some $x \in]a, b[$, aQ^Sx, then $S \in D^s(a, x)$. Thus, by Lemma 3.2 and $a < x < b$, $S \in D(a, b)$ and $\neg bf(R)a$. If for some $x \in]-\infty, a[$, xQ^Sb, then $S \in D^s(x, b)$. Thus, by Lemma 3.2 and $x < a < b$, $S \in D(a, b)$ and $\neg bf(R)a$.

Second, we show (ii). If for some $x \in]a, b[$, $bQ^{S\cup T}x$, then $U \in D^s(b, x)$. Thus, by Lemma 3.2 and $b > x > a$, $U \in D(b, a)$ and $\neg af(R)b$. If for some $x \in]b, +\infty[$, $xQ^{S\cup T}a$, then $U \in D^s(x, a)$. Thus, by Lemma 3.2 and $x > b > a$, $U \in D(b, a)$ and $\neg af(R)b$.

Third, we show (iii). Because the presumptions of (i) and (ii) do not hold, we have

(a) for all $x \in]-\infty, a[$, $\neg xQ^Sb$;

(b) for all $x \in]a, b[$, $\neg aQ^Sx$ and $\neg bQ^{S\cup T}x$; and

(c) for all $x \in]b, +\infty[$, $\neg x Q^{S \cup T} a$.

Let $x \in]-\infty, a[$. We want to show that $\neg x f(R) a$ and $\neg x f(R) b$. Because for all $i \in N$, R_i is single-peaked, we have for all $i \in T \cup U$, $a P_i x$ and $b P_i x$. Let $y \in]x, a[$. By (a), $b Q^S y$. Thus, $T \cup U \in D^s(b, y)$. By Lemma 3.2 and $b > y > x$, $T \cup U \in D(b, x)$. Hence, $\neg x f(R) b$. By Theorem 3.1, $f(R) \in \mathcal{C}$. Thus, from $x < a < b$ and $\neg x f(R) b$ we obtain $\neg x f(R) a$.

Let $x \in]a, b[$. We want to show that $\neg a f(R) x$ and $\neg b f(R) x$. Because for all $i \in N$, R_i is single-peaked, we have for all $i \in S$, $x P_i b$, for all $i \in U$, $x P_i a$, and for all $i \in T$, $x P_i a$ and $x P_i b$. Let $y \in]x, b[$. By (b), $y Q^S a$ and $T \cup U \in D^s(y, a)$. Thus, by Lemma 3.2 and $y > x > a$, $T \cup U \in D(x, a)$ and $\neg a f(R) x$. Let $z \in]a, x[$. By (b), $z Q^{S \cup T} b$ and $S \cup T \in D^s(z, b)$. Thus, by Lemma 3.2 and $z < x < b$, $S \cup T \in D(x, b)$ and $\neg b f(R) x$.

Let $x \in]b, +\infty[$. We want to show that $\neg x f(R) a$ and $\neg x f(R) b$. Because for all $i \in N$, R_i is single-peaked, we have for all $i \in S \cup T$, $a P_i x$ and $b P_i x$. Let $y \in]b, x[$. By (c), $a Q^{S \cup T} y$. Thus, $S \cup T \in D^s(a, y)$. By Lemma 3.2 and $a < y < x$, $S \cup T \in D(a, x)$. Hence, $\neg x f(R) a$. By Theorem 3.1, $f(R) \in \mathcal{C}$. Thus, from $a < b < x$ and $\neg x f(R) a$ we obtain $\neg x f(R) b$. □

In (i) and (ii) of Lemma 3.5, the ranking of $f(R)$ over $\{a, b\}$ is uniquely determined by Q. If neither (i) nor (ii) hold, then the ranking of $f(R)$ over $\{a, b\}$ can be taken arbitrarily without causing intransitivities.

3.3 The Characterization

In the previous subsection we identified a monotone collection Q of strictly quasi-concave orderings from a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*. Theorem 3.1 and Lemma 3.5 imply that the welfare function is completely determined by Q. These results reveal the characteristics of such a welfare function.

In formulating the characterization result, we need a precise definition of tie-breaking. Given $a, b \in \mathbb{R}$, let $\mathcal{W}_{\{a,b\}}$ denote the set of weak orderings over $\{a, b\}$. Obviously, only three orderings over $\{a, b\}$ are possible: a is strictly preferred to b, a and b are indifferent, and b is strictly preferred to a. A tie-breaking rule τ is a family of functions $\tau_{\{a,b\}}$, indexed by $\{a, b\}$ (where $a, b \in \mathbb{R}$ and $a \neq b$), from $\mathcal{W}_{\{a,b\}}^N$ to $\mathcal{W}_{\{a,b\}}$. The function $\tau_{\{a,b\}}$ assigns to each profile of orderings over $\{a, b\}$ an element in $\mathcal{W}_{\{a,b\}}$. Tie-breaking should not be read to be strict because the resulting ordering over $\{a, b\}$ may be indifference.

Fixed-Strictly-Quasi-Concave Welfare Function, f_τ^Q: Given a monotone collection Q of strictly quasi-concave orderings such that $Q^\emptyset = \geq$ and $Q^N = \leq$, and a tie-breaking rule τ, the *fixed-strictly-quasi-concave welfare function* f_τ^Q associated with Q and τ is defined as follows. Let $R \in \mathcal{R}^N$ and $a, b \in \mathbb{R}$ be such that $a \leq b$. Let $S \equiv \{i \in N \mid aP_ib\}$, $U \equiv \{i \in N \mid bP_ia\}$, and $T \equiv N \backslash (S \cup U)$. Then

(α) if $a = b$, then $af_\tau^Q(R)b$;

(β) if $a < b$ and for some $x \in]a, b[$, aQ^Sx, or for some $x \in]-\infty, a[$, xQ^Sb, then $af_\tau^Q(R)b$ and $\neg bf_\tau^Q(R)a$;

(γ) if $a < b$ and for some $x \in]a, b[$, $bQ^{S\cup T}x$, or for some $x \in]b, +\infty[$, $xQ^{S\cup T}a$, then $bf_\tau^Q(R)a$ and $\neg af_\tau^Q(R)b$; and

(δ) if $a < b$ and (a) for all $x \in]-\infty, a[$, $\neg xQ^Sb$, (b) for all $x \in]a, b[$, $\neg aQ^Sx$ and $\neg bQ^{S\cup T}x$, and (c) for all $x \in]b, +\infty[$, $\neg xQ^{S\cup T}a$, then $f_\tau^Q(R)|\{a, b\} \equiv \tau_{\{a,b\}}(R|\{a, b\})$. ∘

Lemma 3.6 *A fixed-strictly-quasi-concave welfare function is a well-defined welfare function satisfying weak Pareto and independence of irrelevant alternatives.*

Proof. Let Q be a monotone collection of strictly quasi-concave orderings and τ be a tie-breaking rule. Let f_τ^Q be defined as above.

First, we prove well-definedness of f_τ^Q. We have to show that (β) and (γ) exclude each other.

Let $a, b \in \mathbb{R}$ be such that (β) holds for a and b. Thus, $a < b$. If for some $x \in]a, b[$, $aQ^S x$, then by strict quasi-concavity of Q^S and $a < x < b$, $\neg bQ^S x$. Thus, by transitivity, $\neg bQ^S a$. If for some $x \in]-\infty, a[$, $xQ^S b$, then by strict quasi-concavity of Q^S and $x < a < b$, $\neg bQ^S a$. Hence, in both cases we have $\neg bQ^S a$ and $aQ^S b$.

Because Q is monotone and $aQ^S b$, by (i.ii) of Lemma 3.4 we have for all $y \in]a, b[$, $\neg bQ^{S\cup T} y$. Because Q is monotone and $aQ^S b$, by (i.i) of Lemma 3.4 we have for all $y \in]b, +\infty[$, $\neg yQ^{S\cup T} a$. Hence, (γ) does not hold.

Second, we show that f_τ^Q is a welfare function. Let $R \in \mathcal{R}^N$. By definition, $f_\tau^Q(R)$ is complete. It remains to show that $f_\tau^Q(R)$ is transitive. Let $a, b, c \in \mathbb{R}$ be such that $a < b < c$. If $\neg a f_\tau^Q(R) b$ and $\neg c f_\tau^Q(R) b$, then $f_\tau^Q(R)|\{a, b, c\}$ is transitive. Thus, in proving transitivity of $f_\tau^Q(R)$, it suffices to show the following two implications: if $a f_\tau^Q(R) b$, then $\neg c f_\tau^Q(R) a$; and if $c f_\tau^Q(R) b$, then $\neg a f_\tau^Q(R) c$.

We only prove the first implication. The second implication follows similarly. Let $a f_\tau^Q(R) b$. Then (β) or (δ) holds for a and b. Let $S \equiv \{i \in N \mid aP_i b\}$, $U \equiv \{i \in N \mid bP_i a\}$, $T \equiv N \backslash (S \cup U)$, and $S' \equiv \{i \in N \mid aP_i c\}$. Because $a < b < c$ and for all $i \in N$, R_i is single-peaked, we have $S \cup T \subseteq S'$.[4] We distinguish two cases.

Case 1: $a f_\tau^Q(R) b$ because of (β).

If for some $x \in]a, b[$, $aQ^S x$, then, because Q is monotone and $S \subseteq S'$, for some $y \in]x, b[$, $aQ^{S'} y$. Since $a < b < c$, we have $y \in]a, c[$ and by (β), $\neg c f_\tau^Q(R) a$. If for some $x \in]-\infty, a[$, $xQ^S b$, then, because Q is monotone and $S \subseteq S'$, for some $y \in]x, a[$, $yQ^{S'} b$. Thus, by strict quasi-concavity of $Q^{S'}$ and $y < a < b < c$, $yQ^{S'} c$. Hence, by (β), $\neg c f_\tau^Q(R) a$.

Case 2: $a f_\tau^Q(R) b$ because of (δ).

We show that (β) holds for a and c. Let $x \in]b, c[$. Then by (δ), part (c), $\neg xQ^{S\cup T} a$. Thus, $aQ^{S\cup T} x$. Because Q is monotone and $S \cup T \subseteq S'$, for some $y \in]x, c[$, $aQ^{S'} y$.

[4]Note that this implication remains true if for all $i \in N$, R_i is strictly quasi-concave.

Thus, by $y \in]a, c[$ and (β), $\neg c f_\tau^Q(R)a$, the desired conclusion.

Third, we show that f_τ^Q satisfies *weak Pareto* and *independence of irrelevant alternatives*. Because $Q^\emptyset = \geq$ and $Q^N = \leq$, it is easy to check that f_τ^Q satisfies *weak Pareto*. Obviously, by definition, f_τ^Q satisfies *independence of irrelevant alternatives*. □

The following theorem characterizes the class of welfare functions satisfying Arrow's requirements on the domain of all single-peaked preferences over a one-dimensional policy space.

Theorem 3.2 *On the domain of single-peaked preferences, the class of fixed-strictly-quasi-concave welfare functions is characterized by* weak Pareto *and* independence of irrelevant alternatives.

Proof. From Lemma 3.6 it follows that a fixed-strictly-quasi-concave welfare function is a well-defined welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*.

Conversely, let f be a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*. Lemma 3.4 shows the existence of a monotone collection Q of strictly quasi-concave orderings. Let $R \in \mathcal{R}^N$ and $a, b \in \mathbb{R}$. From Lemma 3.5 it follows that $f(R)|\{a, b\}$ satisfies the definitions (α), (β), and (γ). Finally, if (δ) holds, then define $\tau_{\{a,b\}}(R|\{a, b\}) \equiv f(R)|\{a, b\}$. Now it is obvious that f is a fixed-strictly-quasi-concave welfare function associated with Q and τ. □

Theorem 3.2 remains valid if the set of alternatives is a open or compact interval and/or each individual's preference domain is the set of all symmetric single-peaked preferences.

3.4 Majority Voting

In this subsection we explore the relation between fixed-strictly-quasi-concave welfare functions and majority voting. Under majority voting, for a given profile of preferences, one alternative is ranked above another if and only if a majority of voters weakly prefers the former alternative to the latter. If the number of individuals is odd, majority voting is a well-defined welfare function (Black, 1948; Arrow, 1951). However, if the number of individuals is even, then the social indifference relation may be intransitive. In resolving this intransitivity, we add an odd number of fixed strictly quasi-concave relations and determine the majority preference relation for the profile of individual's preferences and the fixed voter's preferences.

Let $O \equiv (O_0, O_1, \ldots, O_n)$ denote a profile of $n+1$ fixed voters' preferences such that $O_0 = \geq$, $O_n = \leq$, and for all $t \in \{1, \ldots, n-1\}$, $O_t \in \mathcal{C}$. Let $Q(O)$ denote the collection of strictly quasi-concave preferences associated with O, i.e. for all $S \subseteq N$, $Q^S = O_{|S|}$. We call O *strongly monotone* when for all $S, T \subseteq N$ such that $S \subseteq T$ and all $a, b \in \mathbb{R}$ such that $a < b$, (i) if $aO_{|S|}b$, then $aO_{|T|}b$; and (ii) if $bO_{|T|}a$, then $bO_{|S|}a$. Obviously, if O is strongly monotone, then $Q(O)$ is monotone. Given a strongly monotone profile O of $n+1$ fixed voters, the *majority welfare function* g^O associated with O is defined as follows. For all $R \in \mathcal{R}^N$ and all $a, b \in \mathbb{R}$,

$$\begin{aligned} ag^O(R)b \Leftrightarrow & |\{i \in N \mid \neg bR_i a\}| \cup |\{t \in \{0, 1, \ldots, n\} \mid \neg bO_t a\}| \\ & \geq |\{i \in N \mid \neg aR_i b\}| \cup |\{t \in \{0, 1, \ldots, n\} \mid \neg aO_t b\}|. \end{aligned}$$

The majority tie breaking rule τ^O associated with O is defined as follows: for all $R \in \mathcal{R}^N$ and all $a, b \in \mathbb{R}$, $\tau^O_{\{a,b\}}(R|\{a,b\}) \equiv g^O(R)|\{a,b\}$.

If O is a strongly monotone profile of $n+1$ fixed voters, then the majority welfare function g^O is the fixed-strictly-quasi-concave welfare function associated with $Q(O)$ and the majority tie-breaking rule τ^O.

Proposition 3.1 *If O is a strongly monotone profile of $n+1$ fixed voters, then $g^O = f^{Q(O)}_{\tau^O}$.*

Proof. Let $R \in \mathcal{R}^N$ and $a, b \in \mathbb{R}$ be such that $a < b$. Let $S \equiv \{i \in N \,|\, aP_ib\}$, $U \equiv \{i \in N \,|\, bP_ia\}$, and $T \equiv N \backslash (S \cup U)$. It suffices to show that $f_{\tau^O}^{Q(O)}|\{a,b\} = g^O(R)|\{a,b\}$.

Let $af_{\tau^O}^{Q(O)}(R)b$. If $af_{\tau^O}^{Q(O)}(R)b$ because of (δ), then by definition of τ^O, $ag^O(R)b$. If $af_{\tau^O}^{Q(O)}(R)b$ because of (β), then $\neg bO_{|S|}a$ and because O is strongly monotone, at least $n+1$ voters strictly prefer a to b, namely $S \cup \{|S|, |S|+1, \ldots, |N|\}$. Thus, $\neg bg^O(R)a$. Similarly it can be shown that $bf_{\tau^O}^{Q(O)}(R)a$ implies $bg^O(R)a$.

Let $ag^O(R)b$. First, we show that if $\neg bO_{|S|}a$, then (γ) does not hold. Because O is strongly monotone, we have $\neg bO_{|S\cup T|}a$. Thus, by strict quasi-concavity of $O_{|S\cup T|}$, for all $x \in]a,b[$, $\neg bO_{|S\cup T|}x$, and for all $x \in]b,+\infty[$, $\neg xO_{|S\cup T|}a$, and (γ) does not hold. Thus, if $\neg bO_{|S|}a$, then (β) or (δ) holds for a and b and by definition, $af_{\tau^O}^{Q(O)}(R)b$.

Similarly it can be shown if $bg^O(R)a$ and $\neg aO_{|S\cup T|}b$, then (β) does not hold and $bf_{\tau^O}^{Q(O)}(R)a$.

If $bO_{|S|}a$ and $aO_{|S\cup T|}b$, then (β) and (γ) do not hold and by (δ), $f_{\tau^O}^{Q(O)}|\{a,b\} = g^O(R)|\{a,b\}$, the desired conclusion. □

If f_τ^Q is an *anonymous* welfare function, then for all coalitions S, Q^S depends only on the cardinality of $|S|$. In particular, the profile $(Q_\emptyset, Q_{\{1\}}, Q_{\{1,2\}}, \ldots, Q_N)$ is a profile of $n+1$ fixed voters. However, majority welfare functions are not characterized by *weak Pareto*, *independence of irrelevant alternatives*, and *anonymity*. This is due to the fact that the tie breaking rule τ need not be the majority tie breaking rule associated with the profile of $n+1$ fixed voters.

If n is odd, for all $k \in \{0, 1, \ldots, \frac{1}{2}(n-1)\}$, $O_k = \geq$, and for all $l \in \{\frac{1}{2}(n-1)+1, \ldots, n\}$, $O_l = \leq$, then Theorem 3.1 and Proposition 3.1 yield Black's celebrated median voter theorem saying that the median of the individual peaks is equal to the alternative that is top-ranked according to simple majority preference. Let g^m denote the simple majority welfare function. As an application of our results we show the following. If the number of voters is odd and the policy space is one-dimensional,

then simple majority voting is transitive if and only if each voter's preference relation is strictly quasi-concave.

Theorem 3.3 *Let $|N|$ be odd and $|N| \neq 1$. The domain of strictly quasi-concave preferences is the unique maximal domain $\bar{\mathcal{R}}^N$ such that*

(i) $\mathcal{R}^N \subseteq \bar{\mathcal{R}}^N \subseteq \mathcal{W}^N$ and

(ii) simple majority voting is transitive on the domain $\bar{\mathcal{R}}^N$.

Proof. Obviously, $|N| \geq 3$. It is straightforward to adjust the definition of a fixed-strictly-quasi-concave welfare function to $\mathcal{C}^N$ and to show that Lemma 3.6 remains true. Hence, by Proposition 3.1, simple majority voting is transitive on the domain $\mathcal{C}^N$.

Suppose that $\bar{\mathcal{R}}^N$ is a domain such that $\mathcal{R}^N \subseteq \bar{\mathcal{R}}^N$ and majority voting is transitive. Thus, g^m is a welfare function with domain $\bar{\mathcal{R}}^N$ satisfying *weak Pareto* and *independence of irrelevant alternatives*. Suppose that $\bar{\mathcal{R}} \backslash \mathcal{C} \neq \emptyset$. Let $R_0 \in \bar{\mathcal{R}} \backslash \mathcal{C}$. Define $f : \mathcal{R}^{N \backslash \{n\}} \to \mathcal{W}$ as follows: for all $R \in \mathcal{R}^{N \backslash \{n\}}$, $f(R) \equiv g^m(R, R_0)$. Because g^m is simple majority voting and $|N| \geq 3$, f satisfies *weak Pareto*. Thus, f is a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*. Let $S \equiv \{1, \ldots, \frac{1}{2}(|N| - 1)\}$. By definition of f and the fact that g^m is simple majority voting, we have $\tilde{f}(\leq_S, \geq_{N \backslash (S \cup \{n\})}) = R_0$. Because R_0 is not strictly quasi-concave, this contradicts Lemma 3.4. □

4 Multi-Dimensional Policy Spaces

This section considers environments where the set of alternatives is multi-dimensional and connected. Let $A \subseteq \mathbb{R}^m$ be a set of alternatives such that the relative interior of A in $\mathbb{R}^m$ is non-empty. A preference relation $R_i \in \mathcal{W}$ is *Euclidean* if there exists a *bliss point* $p(R_i) \in A$ such that for all $a, b \in A$, $aR_ib \Leftrightarrow \|a - p(R_i)\| \leq \|b - p(R_i)\|$

(here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^m$). In this section, $\mathcal{R}$ denotes the set of all Euclidean preferences over A.

For the special case of $A = \mathbb{R}^m$, Border (1984) and Le Breton and Weymark (1996) establish Arrow's theorem. For $A = \mathbb{R}^m_+$, Le Breton and Weymark (2000a,b) show Arrow's theorem. However, all these results rely importantly on the unboundedness of the set of alternatives. In real life, policy makers face budget constraints and these results do not apply.

We establish the following theorem: Arrow's requirements are inconsistent if the set of alternatives has a non-empty and connected interior and it is contained in the closure of its interior (i.e. its boundary has no "tails").

Let $\text{int}A$ denote the interior of A in $\mathbb{R}^m$ and $\text{cl}(A)$ the closure of A in $\mathbb{R}^m$.

Theorem 4.1 *Let $A \subseteq \mathbb{R}^m$ be such that $intA \neq \emptyset$, $intA$ is connected, and $A \subseteq cl(intA)$. If $m \geq 2$, then weak Pareto, independence of irrelevant alternatives, and non-dictatorship are inconsistent on the domain $\mathcal{R}^N$.*

In particular, Theorem 4.1 yields Arrow's theorem for compact and convex sets of policy alternatives with non-empty interior.

If the interior of the set of alternatives is non-empty and not connected, then an Arrovian social welfare function may not be *dictatorial*.

Example 4.1 Let $m \equiv 2$, $A_1 \equiv [0,1] \times [0,1]$, $A_2 \equiv [0,1] \times [9,10]$, and $A \equiv A_1 \cup A_2$. For all $R \in \mathcal{R}^N$, let $\varphi(R)|A_1 = R_1|A_1$, $\varphi(R)|A_2 = R_2|A_2$, and for all $a \in A_1$ and all $b \in A_2$, $\varphi(R)|\{a,b\} = R_1|\{a,b\}$. Then φ is an Arrovian welfare function satisfying *non-dictatorship*. $\lhd$

The conclusion of Example 4.1 does not remain true if the sets A_1 and A_2 are connected through a line segment, for example if $A = A_1 \cup A_2 \cup [(0,0),(0,10)]$. For this set of alternatives an Arrovian welfare function must be dictatorial.[5]

[5]By Theorem 4.1, $f|A_1$ and $f|A_2$ are dictatorial. Because A_1 and A_2 are connected through the line segment $[(0,0),(0,10)]$ it follows from Theorem 3.2 that the dictator of $f|A_1$ and $f|A_2$ is the same and f is dictatorial.

If the boundary of A contains tails, then again an Arrovian welfare function may not be *dictatorial*.

Example 4.2 Let $m \equiv 2$, $N \equiv \{1,2\}$, $B \equiv [0,1] \times [0,1]$, and $L \equiv](0,1),(0,2)]$. Let $A \equiv B \cup L$. For all $R \in \mathcal{R}^N$ and all $a, b \in A$, (i) if $a, b \in L$, $a_2 < b_2$, and for some $i \in N$, aP_ib, then $\neg bf(R)a$, and (ii) otherwise, $f(R)|\{a,b\} = R_1|\{a,b\}$. We show that f is a welfare function.

Claim: For all $R \in \mathcal{R}^N$, $f(R)$ is transitive.
Proof of Claim. If $p(R_1) \in B$, then $f(R) = R_1$. Let $p(R_1) \in L$ and $a, b, c \in A$. If $a, b, c \in B$ or $a, b, c \in L$, then it is easy to check that $f(R)|\{a,b,c\}$ is transitive. If $a \in L$ and $b, c \in B$, then $f(R)|\{a,b,c\} = R_1|\{a,b,c\}$ and $f(R)|\{a,b,c\}$ is transitive. Let $a, b \in L$ and $c \in B$. Then $f(R)|\{a,c\} = R_1|\{a,c\}$ and $f(R)|\{b,c\} = R_1|\{b,c\}$. Obviously, if $f(R)|\{a,b\} = R_1|\{a,b\}$, then $f(R)|\{a,b,c\}$ is transitive. Let $f(R)|\{a,b\} \neq R_1|\{a,b\}$. Without loss of generality, suppose that $a_2 < b_2$. Thus, by (i), $\neg bf(R)a$, and $a_2 \leq p_2(R_1)$. Then aP_1c and $\neg cf(R)a$. Hence, $f(R)|\{a,b,c\}$ is transitive. ◇

Now it is easy to check that f is a welfare function satisfying *weak Pareto*, *independence of irrelevant alternatives*, and *non-dictatorship*. ◁

4.1 Proof of Theorem 4.1

Let[6] f be a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*. Given $X \subseteq A$, let f^X denote the restriction of f to the set of alternatives X. Formally, for all $R \in \mathcal{R}^N$ such that for all $i \in N$, $p(R_i) \in X$, let $f^X(R|X) \equiv f(R)|X$. It is straightforward that f^X inherits *weak Pareto* and *independence of irrelevant alternatives* from f.

[6]The first version of the paper used Theorem 3.2 to prove Theorem 4.1 for the case when A is compact and convex. A proof using Arrow's theorem was suggested by John Duggan to the first author while he was visiting University of Rochester. Independently the second author established the proof below.

We introduce additional notation. Let $a, b, c \in \mathbb{R}^m$ be three distinct alternatives. Let $[a, b]$ denote the line segment between a and b, i.e. $[a, b] \equiv \{\lambda a + (1 - \lambda)b \mid \lambda \in [0, 1]\}$. Let $H(a, b)$ denote the hyperplane that bisects $[a, b]$ perpendicularly. In the non-degenerate case, that is a, b, and c are non-colinear, the hyperplanes $H(a, b)$, $H(a, c)$, and $H(b, c)$ determine six disjoint open cones, say $\angle(abc)$, $\angle(acb)$, $\angle(bac)$, $\angle(bca)$, $\angle(cab)$, and $\angle(cba)$ (see Figure 1). We named these cones such that for all $R_i \in \mathcal{R}$, $p(R_i) \in \angle(acb) \Leftrightarrow aP_icP_ib$. The same holds for the other cones. Note that the collective boundary of $\angle(abc)$ and $\angle(acb)$ is a subset of $H(b, c)$. Let $\angle(a(bc))$ denote the set of all points $x \in H(b, c)$ such that for $R_i \in \mathcal{R}$, if $p(R_i) = x$, then aP_ibI_ic.

Given $a \in \mathbb{R}^m$ and $\epsilon > 0$, let $\bigodot(a, \epsilon)$ denote the open ball with center a and radius ϵ, i.e. $\bigodot(a, \epsilon) \equiv \{x \in \mathbb{R}^m \mid \|a - x\| < \epsilon\}$. Given $a, b, c \in \mathbb{R}^m$, let $\text{plane}(a, b, c)$ denote the plane spanned by a, b, and c, i.e. $\text{plane}(a, b, c) \equiv \{a + \lambda(b - a) + \mu(c - a) \mid \lambda, \mu \in \mathbb{R}\}$. Finally, let $\triangle(a, b, c)$ denote the triangle with corner points a, b, and c, i.e. $\triangle(a, b, c) \equiv \{\lambda a + \mu b + (1 - \lambda - \mu)c \mid \lambda, \mu \geq 0 \,\&\, \lambda + \mu \leq 1\}$. The baricenter of $\triangle(a, b, c)$ is the point $\frac{1}{3}(a + b + c)$.

The following lemma generalizes Lemma 3.2.

Lemma 4.1 *Let $S \subseteq N$ and $a, b \in A$ be such that $a \neq b$. Let $c \in A \backslash \{a, b\}$ be such that $\angle(abc) \cap A \neq \emptyset$, $\angle(bac) \cap A \neq \emptyset$, $\angle(b(ac)) \cap A \neq \emptyset$, and $\angle(bca) \cap A \neq \emptyset$. If $S \in D^s(a, b)$, then $S \in D(a, c)$.*

Proof. Let $R \in \mathcal{R}^N$ be such that for all $i \in S$, aP_ic. Let $S' \equiv \{i \in N \mid aP_ic\}$, $T \equiv \{i \in N \mid aI_ic\}$, and $V \equiv \{i \in N \mid cP_ia\}$. Because $\angle(abc)$, $\angle(bac)$, $\angle(b(ac))$, and $\angle(bca)$ have a non-empty intersection with A, there exists $\tilde{R} \in \mathcal{R}^N$ such that for all $i \in S$, $a\tilde{P}_ib\tilde{P}_ic$, for all $i \in S' \backslash S$, $b\tilde{P}_ia\tilde{P}_ic$, for all $i \in T$, $b\tilde{P}_ia\tilde{I}_ic$, and for all $i \in V$, $b\tilde{P}_ic\tilde{P}_ia$. By $S \in D^s(a, b)$ and *independence of irrelevant alternatives*, $af(\tilde{R})b$. By *weak Pareto*, $\neg cf(\tilde{R})b$. Thus, by transitivity of $f(\tilde{R})$, $\neg cf(\tilde{R})a$. Hence, by *independence of irrelevant alternatives*, $\neg cf(R)a$ and $S \in D(a, c)$, the desired conclusion. $\square$

Note that if A is the real line, then the assumptions of Lemma 4.1 only hold if $a < b < c$ or $a > b > c$. While in the one-dimensional case, Lemma 4.1 implies only a limited decisiveness power, in the multi-dimensional case decisiveness appears to be unbounded.

For the moment, suppose that A is convex. Let $a, b, c \in A$ be three distinct alternatives. Then $\angle(abc) \cap A \neq \emptyset$, $\angle(bac) \cap A \neq \emptyset$, $\angle(b(ac)) \cap A \neq \emptyset$, and $\angle(bca) \cap A \neq \emptyset$ whenever $\angle(abc) \cap A \neq \emptyset$ and $\angle(bca) \cap A \neq \emptyset$. Because A is convex $\angle(abc) \cap A \neq \emptyset$ if the baricenter of $\triangle(a, b, c)$ belongs to the same half plane to which c belongs induced by the line containing $[a, b]$ in plane(a, b, c). This means that $\triangle(a, b, c)$ is acute at c. Similarly $\angle(bca) \cap A \neq \emptyset$ if $\triangle(a, b, c)$ is acute at a. This is accomplished if c belongs to $Y(a, b) \equiv \{x \in A \setminus \bigodot(\frac{1}{2}(a + b), \frac{1}{2}\|a - b\|) \mid x$ is in the same open halfspace as b, where this halfspace is induced by the hyperplane through a perpendicular to $[a, b]\}$ (see Figure 2).

These considerations together with Lemma 4.1 yield the following.

Lemma 4.2 *Let A be convex and $a, b \in A$ be such that $a \neq b$. If $S \in D^s(a, b)$, then for all $c \in Y(a, b)$, $S \in D(a, c)$.*

The following lemma expands this decisiveness power even further if $\bigodot(a, \|a-b\|)$ is a subset of A. Given a set $X \subseteq \mathbb{R}^m$, let $\partial X \equiv \mathrm{cl}(X) \setminus \mathrm{int} X$ denote the boundary of X.

Lemma 4.3 *Let A be convex and $a, b \in A$ be such that $a \neq b$ and $\bigodot(a, \|a-b\|) \subseteq A$. If $S \in D^s(a, b)$, then for all $c \in A \setminus \{a\}$, $S \in D(a, c)$.*

Proof. Note that by $\bigodot(a, \|a-b\|) \subseteq A$, $\cup_{x \in \partial \bigodot(a, \|a-b\|)} Y(a, x) = A \setminus \{a\}$. By Lemma 4.2, for all $c \in Y(a, b) \cap \partial \bigodot(a, \|a - b\|)$, $S \in D(a, c)$. Because $S \in D(a, c)$ implies $S \in D^s(a, c)$, Lemma 4.2 implies that for all $c \in Y(a, b) \cap \partial \bigodot(a, \|a - b\|)$ and all $c' \in Y(a, c)$, $S \in D(a, c')$. Hence, for all $c \in A \setminus \{a\}$, $S \in D(a, c)$. □

Next by applying Lemma 4.3 and Arrow's theorem we show that if the set of alternatives is an open ball, then f is dictatorial.

Lemma 4.4 *Let $x \in \mathbb{R}^m$, $\epsilon > 0$, and $A = \bigodot(x, \epsilon)$. Then f is dictatorial.*

Proof. Let $a, b \in \bigodot(x, \epsilon)$ be such that $a \neq b$. Because $a, b \notin \partial \bigodot(x, \epsilon)$, there exists $c \in \bigodot(x, \epsilon)$ such that $\triangle(a, b, c)$ is acute. But then the baricenter of $\triangle(a, b, c)$ belongs to $\triangle(a, b, c)$. Hence, for all $u, v, w \in \{a, b, c\}$ such that $\{u, v, w\} = \{a, b, c\}$, $\angle(uvw) \cap \bigodot(x, \epsilon) \neq \emptyset$.[7] Applying Arrow's theorem yields that $f^{\{a,b,c\}}$ is dictatorial with dictator $i_{\{a,b,c\}}$. Since $c \in \bigodot(x, \epsilon)$, there exists $\epsilon' > 0$ such that $\bigodot(c, \epsilon') \subseteq \bigodot(x, \epsilon)$. Now we can find $a', b' \in \bigodot(c, \epsilon')$ such that $\triangle(a', b', c)$ is acute and $\bigodot(c, \|a' - c\|) \subseteq \bigodot(c, \epsilon')$. Similarly to $\triangle(a, b, c)$, Arrow's theorem implies that $f^{\{a',b',c\}}$ is dictatorial with dictator $i_{\{a',b',c\}}$. Because $\{i_{\{a',b',c\}}\} \in D(c, a')$ and $\bigodot(c, \|a' - c\|) \subseteq \bigodot(x, \epsilon)$, Lemma 4.3 implies that for all $y \in \bigodot(x, \epsilon) \backslash \{c\}$, $\{i_{\{a',b',c\}}\} \in D(c, y)$. Therefore, $\{i_{\{a',b',c\}}\} \in D(c, a)$ and $\{i_{\{a,b,c\}}\} \in D(c, a)$. Thus, $i_{\{a',b',c\}} = i_{\{a,b,c\}}$.

Hence, we have shown that for all $a, b \in \bigodot(x, \epsilon)$ such that $a \neq b$, there exists $c \in \bigodot(x, \epsilon)$ and $i_{\{a,b,c\}} \in N$ such that $i_{\{a,b,c\}}$ is decisive over "a preferred to b", over "b preferred to a", over "c preferred to a", and for all $y \in \bigodot(x, \epsilon) \backslash \{c\}$, over "$c$ preferred to y". Clearly it follows that $i_{\{a,b,c\}}$ is the same individual for all those $a, b, c \in \bigodot(x, \epsilon)$. Let i denote this individual. Thus, for all $a, b \in \bigodot(x, \epsilon)$ such that $a \neq b$, i is decisive over "a preferred to b". Hence, f is dictatorial. □

Next we show that f^{intA} is dictatorial if $\mathrm{int}A$ is connected.

Lemma 4.5 *Let $intA$ be non-empty and connected. Then f^{intA} is dictatorial.*

Proof. First, note that for all $x \in \mathrm{int}A$ there is some $\epsilon_x > 0$ such that $\bigodot(x, \epsilon_x) \subseteq \mathrm{int}A$. Hence, by Lemma 4.4, $f^{\bigodot(x,\epsilon_x)}$ is dictatorial, say with dictator i_x.

Let $a, b \in \mathrm{int}A$ be such that $a \neq b$. It is sufficient to show that $i_a = i_b$ and i_a is decisive over "a preferred to b".

[7] In other words, the set $\{a, b, c\}$ is a free triple.

Because int$A$ is connected, there is a path in int$A$ from a to b. Clearly, the path is compact and therefore, there exists a sequence $c_0, c_1, \ldots, c_k$ on this path and $\epsilon > 0$ such that (i) $c_0 = a$ and $c_k = b$, (ii) for all $t \in \{0, 1, \ldots, k\}$, $\bigodot(c_t, \epsilon) \subseteq \text{int}A$, (iii) for all $t \in \{1, \ldots, k\}$, $\bigodot(c_{t-1}, \epsilon) \cap \{c_t, c_{t+1}, \ldots, c_k\} = \{c_t\}$. For all $t \in \{0, 1, \ldots, k\}$, let i_t denote the dictator of $f^{\bigodot(c_t,\epsilon)}$.

By (i), (ii), and (iii), it follows that $i_a = i_0 = i_1 = \cdots = i_k = i_b$. Thus, $i_a = i_b$. For all $t \in \{1, \ldots, k-1\}$, let $R^t_{i_a} \in \mathcal{R}$ be such that $p(R^t_{i_a}) = c_{t-1}$. Thus, by (iii), $c_{t-1} P^t_{i_a} c_t P^t_{i_a} b$. Using these preferences, the fact that $i_a = i_{k-1} = i_b$, (iii), and transitivity, it follows that i_a is decisive over "c_{k-2} preferred to b". Then, by $i_a = i_{k-2} = i_b$ and the previous fact, i_a is decisive over "c_{k-3} preferred to b". Continuing in this way it follows that for all $t \in \{1, \ldots, k\}$, i_a is decisive over "c_{k-t} preferred to b". Hence, i_a is decisive over "a preferred to b", the desired conclusion. □

Our final lemma completes the proof of Theorem 4.1. If $A \backslash \text{int}A$ contains no "tails", i.e. $A \subseteq \text{cl}(\text{int}A)$, then dictatorship on the interior of A expands to the whole set A.

Lemma 4.6 *Let intA be non-empty and connected, and $A \subseteq$ cl(intA). If f^{intA} is dictatorial, then f is dictatorial.*

Proof. Without loss of generality, let 1 be the dictator of f^{intA}. We show that 1 is the dictator of f.

Let $a, b \in A$ be such that $a \neq b$. Let $R \in \mathcal{R}^N$ be such that aP_1b. It suffices to show that $\neg b f(R) a$.

Let $S \equiv \{i \in N \mid aP_ib\}$, $T \equiv \{i \in N \mid aI_ib\}$, and $V \equiv \{i \in N \mid bP_ia\}$. Because $A \subseteq \text{cl}(\text{int}A)$ and int$A$ is connected, there exists a path from a to b in int$A$ such that a and b are the only possible points on this path belonging to $A \backslash$ int$A$. Thus, int$A \cap H(a, b) \neq \emptyset$. Let $h \in \text{int}A \cap H(a, b)$ be such that $h \neq \frac{1}{2}(a + b)$ and $R' \in \mathcal{R}^N$ be such that for all $i \in S$, $p(R'_i) = a$, for all $i \in T$, $p(R'_i) = h$, and for all $i \in V$,

$p(R'_i) = b$. Thus, by $R'|\{a,b\} = R|\{a,b\}$ and *independence of irrelevant alternatives*,

$$f(R)|\{a,b\} = f(R')|\{a,b\}. \tag{2}$$

We consider four cases.

Case 1: $a, b \in \text{int}A$.

Then for all $i \in N$, $p(R'_i) \in \text{int}A$. Because f^{intA} is dictatorial with dictator 1 and $1 \in S$, it follows that $\neg bf(R')a$. Hence, by (2), $\neg bf(R)a$, the desired conclusion.

Case 2: $a \in \text{int}A$ and $b \in A \setminus \text{int}A$.

Because $a \in \text{int}A$, for some $0 < \epsilon < \frac{1}{4}\|a-b\|$ we have $\bigodot(a,\epsilon) \subseteq \text{int}A$. Let $c \in \bigodot(a,\epsilon) \cap \text{plane}(a,b,h) \cap \partial\bigodot(\frac{1}{2}(a+b), \frac{1}{2}\|a-b\|)$ be such that $\|h-c\| < \|h-a\|$. Because $h \in \text{int}A \cap H(a,b)$ and by our choice of c, $h \in \partial\angle(c(ba))$. Thus, $\angle(cba) \cap \text{int}A \neq \emptyset$. Let $R'' \in \mathcal{R}^N$ be such that for all $i \in S \cup T$, $R''_i = R'_i$, and for all $i \in V$, $p(R''_i) \in \angle(cba) \cap \text{int}A$. Then by construction we have for all $i \in S$, $aP''_i cP''_i b$, for all $i \in T$, $cP''_i aI''_i b$, and for all $i \in V$, $cP''_i bP''_i a$. In particular we have $R''|\{a,b\} = R'|\{a,b\} = R|\{a,b\}$. By $a, c \in \text{int}A$ and $aP''_1 c$, Case 1 yields $\neg cf(R'')a$. By *weak Pareto*, $\neg bf(R'')c$. Thus, by transitivity of $f(R'')$, $\neg bf(R'')a$. Hence, by *independence of irrelevant alternatives*, $f(R'')|\{a,b\} = f(R)|\{a,b\}$ and $\neg bf(R)a$, the desired conclusion.

Case 3: $a \in A \setminus \text{int}A$ and $b \in \text{int}A$.

By taking c close to b in the interior of A such that $\triangle(a,b,c)$ is rightangeled at c and h is closer to b than to c and observing that $\angle(bac) \cap \text{int}A \neq \emptyset$ because $h \in \angle((ba)c)$ is on the boundary, we deduce similarly to Case 2 that $\neg bf(R)a$.

Case 4: $a, b \in A \setminus \text{int}A$.

Because $A \subseteq \text{cl}(\text{int}A)$, we can find $c \in \text{int}A$ such that $\|c-a\| < \|c-b\|$. By Cases 2 and 3 and transitivity of $f(R')$ it follows that $\neg bf(R')a$. Hence, by (2), $\neg bf(R)a$, the desired conclusion. □

5 Conclusion

In this paper we solve a classical open problem in Arrovian social choice. For a one-dimensional policy space and the domain of single-peaked preferences, we characterize the social welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives*. As a corollary we obtain that majority voting is transitive if and only if the number of individuals is odd and individual preferences are strictly quasi-concave.

A parallel line of research considers social choice functions. In this case, the objective is to choose for each admissible set, called an agenda, the socially optimal alternatives in this set. *Arrow's choice axiom* requires that if for an admissible set and some admissible subset of it, some choices made at the former set belong to the smaller set, then the choices at the latter set are exactly the choices made at the former set that belong to the smaller set (Arrow, 1959).[8] In spatial environments with one-dimensional policy spaces and agenda domain consisting of all compact intervals, the social choice functions satisfying *weak Pareto*, *Arrow's choice axiom*, and *independence of infeasible alternatives* have been investigated (Moulin, 1984; Ehlers, 2001). These papers obtain characterizations of the "generalized median rules" by these and additional axioms. A generalized median rule assigns to each compact interval and each profile the point belonging to the compact interval that is closest to the median of the peaks reported by the voters and 2^N fixed voters. The quasi bliss points of the 2^N fixed strictly quasi-concave preferences play the role of these fixed voters. The quasi bliss point of the social ordering chosen by an Arrovian welfare function for a profile of preferences is the outcome of the generalized median rule applied to this profile of quasi bliss points. However, the natural correspondence between generalized median rules and Arrovian welfare functions fails because the median may *not* be the socially most preferred alternative. The class of welfare functions underlying median rules is smaller than the class of Arrovian welfare functions: in the former the social

[8]Dutta, Jackson, and Le Breton (2001) consider Arrovian-type axioms in elections in which candidates have the possibilty to withdraw. See also Ehlers and Weymark (2001).

ordering is single-peaked.

The same applies to the strand of literature on strategy-proof social choice rules when the set of alternatives is one-dimensional (Moulin, 1980; Border and Jordan, 1983; Kim and Roush, 1984; Barberà, Gul, and Stacchetti, 1993). For the case of a finite set of alternatives and unrestricted domain, Satterthwaite (1975) associates with each social choice rule satisfying strategy-proofness and unanimity an Arrovian welfare function. For his construction it is essential that any pair of alternatives can be moved to the top two positions in each agent's preference relation. It is obvious that this trick does not work in spatial environments.

For the case when preferences are Euclidean and the set of alternatives is $\mathbb{R}^m$ or $\mathbb{R}^m_+$ (where $m \geq 2$), Arrow's theorem has been previously established. However, these proofs do not apply for instance to the triangle with endpoints $(0,0)$, $(1,0)$, and $(0.5,0.1)$. Then it is not possible to find a point x belonging to this triangle such that $(0,0)$, x, and $(1,0)$ are a free triple. We are the first to show Arrow's theorem when the set of alternatives is compact and convex. Here, we could have tried to follow another classical method of proof. From a social welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives* we could have constructed a two-agents social choice function C with compact and convex agenda domain (including line segments) as follows: fix two coalitions S and $N\backslash S$ (where $S \neq \emptyset$ and $S \neq N$) and consider only profiles where all agents belonging to S have the same preference relation and all agents belonging to $N\backslash S$ have the same preference relation. For each compact and convex agenda A' and each profile $(R_S, R_{N\backslash S}) \in \mathcal{R}\times\mathcal{R}$ (here S denotes "agent 1" and $N\backslash S$ "agent 2"), let C choose the quasi bliss point of $f(R_S, R_{N\backslash S})|[p(R_S|A'), p(R_{N\backslash S}|A')]$. Since at profiles with only two peaks, the *weak Pareto* set in A' is the line segment $[p(R_S|A'), p(R_{N\backslash S}|A')]$ it is easy to see that C satisfies *weak Pareto*, *Arrow's choice axiom*, and *independence of infeasible alternatives*. Then we could apply Duggan's (1996) theorem for spatial environments saying that C must be dictatorial with dictator S or $N\backslash S$.[9] However, Duggan's

[9]Duggan's (1996) theorem does not hold if the agenda domain consists only of non-degenerate

theorem requires the preference domain to contain non-Euclidean preferences and our impossibility result does not.

compact and convex sets and preferences are Euclidean. Then Bergson-Samuelson welfare functions are Arrovian (Le Breton and Weymark, 2000b).

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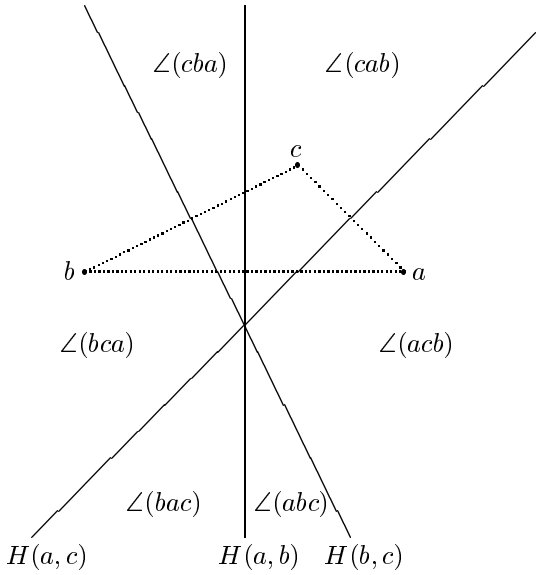

Figure 1: Illustration of the sets $\angle(abc)$, $\angle(acb)$, $\angle(bac)$, $\angle(bca)$, $\angle(cab)$, and $\angle(cba)$ for non-colinear a, b, and c in $\mathbb{R}^2$.

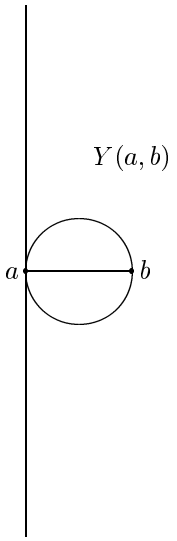

Figure 2: If $A = \mathbb{R}^2$, then $Y(a,b)$ is the half space right of the perpendicular to $[a,b]$ through a excluding the set $\bigodot(\frac{1}{2}(a+b), \frac{1}{2}\|a-b\|)$.