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CAHIER 02-2001

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February 2001

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RÉSUMÉ

Cet article est un survol de la littérature consacrée aux problèmes de l'ordonnancement d'ensembles d'objets. De tels ordres apparaissent dans la théorie des choix en incertitude complète, la théorie des ensembles d'options, la théorie de l'appariement et celle des préférences d'assemblées. Le survol est préparé pour le *Handbook of Utility Theory*, vol. 2, sous la direction de Salvador Barberà, Peter Hammond et Christian Seidl, à paraître chez Kluwer Academib Publishers. Le numéro de chapitre est provisoire.

Mots clés : choix en incertitude complète, ensembles d'option, appariement, préférences d'assemblées

ABSTRACT

We provide a survey of the literature on ranking sets of objects. The interpretations of those set rankings include those employed in the theory of choice under complete uncertainty, rankings of opportunity sets, set rankings that appear in matching theory, and the structure of assembly preferences. The survey is prepared for the *Handbook of Utility Theory*, vol. 2, edited by Salvador Barberà, Peter Hammond, and Christian Seidl, to be published by Kluwer Academic Publishers. The chapter number is provisional.

Key words : choice under complete uncertainty, opportunity sets, matching, assembly preferences

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RANKING SETS OF OBJECTS

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1 INTRODUCTION

Why are we interested in ranking sets of objects? How do we actually rank them? How is a ranking of sets related to a ranking of the primitive objects, provided there is such an underlying ranking? These are the basic issues addressed in this chapter, and in order to motivate their study, consider the following simple example. Suppose an object x is preferred to another object y. Does this tell us anything about the relative rankings of the sets $\{x\}$, $\{y\}$, and $\{x, y\}$, and if so, what does it tell us exactly? The answer to this question (and others of that kind) depends, of course, on the interpretation that we attach to the sets $\{x\}$, $\{y\}$, and $\{x, y\}$.

We will argue in this introduction that any one of six possible rankings of $\{x\}$, $\{y\}$, and $\{x, y\}$ could result, and each one of them may be defended through at least one, and sometimes several, interpretations of our sets of objects. Moreover, each of these interpretations has been examined in connection with problems that arise naturally in economics, and in other fields of knowledge as well. Hence, our tentative answer to the questions opening this introduction is that sets are ranked because this is a way to model and to analyze many interesting problems, and that how they are ranked depends on the specific issue being addressed. Let us now turn to some illustrations.

Example 1 We first consider a scenario where, based on the assumption that x is better than y, $\{x\}$ is preferred to $\{x, y\}$ and $\{x, y\}$ is preferred to $\{y\}$. This is a very natural conclusion under the following interpretation. The objects x and y are mutually exclusive alternatives: only one of the two can materialize. The sets $\{x\}$ and $\{y\}$ are interpreted as situations where the respective alternative occurs with certainty and, because the set $\{x, y\}$ cannot be interpreted as a possible choice under this interpretation, it is taken to represent the possibility that either x or y may be the final outcome. Moreover, suppose that the final choice between x and y is left to a random device. We are not specific about this device, except that it must attach positive probability to both x and y: $\{x, y\}$ is then the support of a lottery with prizes x and y. Under this interpretation, $\{x\}$ is preferred to $\{x, y\}$ which, in turn, is better than $\{y\}$ for any expected-utility maximizer, whatever the exact lottery underlying $\{x, y\}$ might be.

Even if x and y are interpreted as compatible options rather than mutually exclusive alternatives, the above ranking may result. For example, they could represent candidates for a committee of variable size. In that case, an agent who prefers x to y will naturally prefer a committee consisting of x only to a committee of which y is the only member. Furthermore, if x is considered to be a desirable candidate and y is undesirable, having y in addition to x could plausibly be considered worse than having x alone and, by the same token, adding x to $\{y\}$ would be desirable from the viewpoint of someone with these preferences.

Under the first interpretation of this example, the ranking described above is very compelling once we know that x is considered better than y. The same ranking of sets is plausible under the second interpretation, but other possibilities are equally plausible, depending on whether x and y are desirable; see Examples 4 and 5 below.

Example 2 Suppose $\{x\}$ is preferred to $\{x,y\}$ and to $\{y\}$, and $\{x,y\}$ and $\{y\}$ are indifferent. This ranking arises naturally under the interpretation of the sets $\{x\}$, $\{y\}$, and $\{x,y\}$ provided in the first example, where x and y are mutually exclusive alternatives and the final choice of an alternative is left to a random device. Here, instead of the expected-utility criterion, suppose the agent uses the maximin criterion—sets of possible outcomes are ranked on the basis of the worst-case scenario. This extremely pessimistic attitude towards choice under uncertainty leads to the ranking described above.

Example 3 Suppose $\{x\}$ and $\{x,y\}$ are indifferent, and both of those sets are preferred to $\{y\}$. Again, this ranking is compatible with modified versions of our preceding story. As before, suppose x and y are mutually exclusive alternatives, the sets $\{x\}$, $\{y\}$, and $\{x,y\}$ are interpreted as sets of possible final outcomes, and the final choice is performed by means of a random device. If the decision maker under consideration is extremely optimistic and uses the maximax criterion to assess sets of possible outcomes, the above ranking emerges.

Alternatively, change the above interpretation regarding how the final choice is made by assuming that it is the decision maker rather than a random device who determines the final outcome. For example, we can think of $\{x,y\}$ as the set of alternatives that have not been 'screened out' by a previous process, out of which a selection must be made. In this case, it is natural to conclude that the agent is indifferent between choosing from $\{x,y\}$ and choosing from $\{x\}$, provided the decision maker only cares about the final choice itself and not about the opportunities available. This is also known as the indirect-utility criterion: the criterion to rank sets is the best possible choice that can be made or, equivalently, if the preferences that are used to rank the alternatives are representable by a real-valued utility function, the highest possible attainable level of utility. The qualification that the agent only cares about the final choice in this interpretation is relevant at this point, where we introduce a fourth possibility.

Example 4 Suppose now that $\{x,y\}$ is better than $\{x\}$ which, in turn, is better than $\{y\}$. If the agent has the final choice from a set of options but, instead of being exclusively concerned with the quality of the final choice, the decision maker also attaches value to the freedom of choice associated with the set of opportunities from which to choose, this is a plausible ranking.

Our second interpretation of this ranking appeals to different principles and is formalized through quite different models. Rather than the freedom of choice, what might be appreciated from $\{x,y\}$ is that it provides more flexibility than the two singletons $\{x\}$ and $\{y\}$. Suppose, again, that the agent will make a final choice, but not at the time the ranking is established. The set under consideration is interpreted as the set of options it will have in the future. Assume that, even if x is preferred to y today, the agent cannot completely discard the possibility that its preferences might be reversed by the time a choice has to be made, or that one of the alternatives might no longer be available. Then it is natural to prefer $\{x,y\}$ to $\{x\}$ (and to $\{y\}$) because the larger set entails the possibility to change preferences and still be able to choose the best alternative in the end, as well as the possibility of still making a choice, even if one alternative ceases to be available.

Yet another scenario could be invoked to provide a justification of the same ranking. Consider, again, the case where the alternatives x and y are compatible and that both candidates x and y are desirable. Under this interpretation, having both of them is likely to be considered even more desirable than having either one of them separately.

Example 5 Suppose $\{x\}$ is preferred to $\{y\}$ and $\{y\}$ is preferred to $\{x,y\}$. Again, we can appeal to the case where x and y are compatible options, where both of them are undesirable, y more so than x. Getting both of them jointly could then naturally be considered even worse than getting any one of them by itself.

In all of the examples introduced so far, it is the case that x is better than y and the set $\{x\}$ is preferred to the set $\{y\}$. Because of that feature, all the rankings of the sets $\{x\}$, $\{y\}$, and $\{x, y\}$ considered in Examples 1 to 5 are extensions of the ranking of the objects x and y. In the sections to come, we shall often be interested in extensions of a ranking of objects to sets of objects. But this extension requirement will not always be imposed even if it is natural under the interpretation considered (in many situations, it is implied by other assumptions), or it may not be considered suitable at all. The following example discusses a ranking of sets which is not an extension of a ranking of objects.

Example 6 Suppose now that $\{x, y\}$ is preferred to $\{x\}$ and to $\{y\}$, but $\{x\}$ and $\{y\}$ are indifferent, even though x is considered better than y. This ranking could result if the objective is to rank these sets only from the viewpoint of the freedom of choice they offer, without taking into consideration the relative desirability of the final choice.

All of the examples discussed above share the characteristic that no pair composed of any two out of the three sets considered remains unranked, and the ranking thus established is transitive. This will be the case throughout this chapter for all simple comparisons that merely involve two singletons and the two-element set consisting of the constituent elements of those singletons. However, if comparisons of more complex sets have to be made, it may very well be the case that the resulting ranking is not an ordering, either because it is incomplete or because it is not transitive. We will come across examples of that nature later on in this chapter.

2 AN OVERVIEW

As illustrated in the introduction, there are a variety of interpretations that can be given to rankings of sets of alternatives and, depending on the interpretation chosen, different kinds of axioms are considered suitable for being imposed on the rankings in question. At this point, we want to put the topic of this chapter in perspective by discussing some fields of economic analysis where the need to be explicit about rankings of sets has arisen.

The analysis of voting procedures is one such field. The much celebrated Gibbard-Satterthwaite theorem (see Gibbard [62] and Satterthwaite [129]) establishes that only trivial social choice functions can be strategy-proof (in the sense that agents cannot benefit from misrepresenting their preferences) if defined over a universal domain of preferences. In this contex, we use the term 'trivial' to refer to dictatorial rules or rules that are a priori restricted to choosing only among two options. But this result hinges on a definition of manipulability that cannot be properly extended to analyze social choice correspondences unless preferences on the sets that may be chosen are well-defined. The Gibbard-Satterthwaite theorem generalized the understanding that most voting procedures are vulnerable to manipulability and gave a definite confirmation to related conjectures by Dummet and Farquharson [43] and Vickrey [146]. Yet, its robustness has often been questioned because it is formulated in terms that do not allow for multiple choices of alternatives, not even in cases where considerations of symmetry make such choices quite compelling. Work on the extension of the concept of strategy-proofness to multivalued choices led naturally to the examination of the different meanings that can be attached to choosing a set of mutually exclusive objects, and the consequences of each interpretation on how sets are ranked. Early work on this subject is reported, for example, in Barberà [10], Fishburn [50], Gärdenfors [60, 61], Kelly [79], and Pattanaik [111]. More recent work is due to Barberà, Dutta, and Sen [12], Benoît [20], Ching and Zhou [35], Duggan and Schwartz [42], and Pattanaik [111].

A second field where the need to rank sets arises naturally is that of matchings and assignments. For example, firms often hire several workers, colleges admit sets of students, hospitals accept groups of interns, and many schools offer lists of optional courses to students who then, in turn, choose to join their fellow students in some courses but not in others. Allocations of workers to firms, students to colleges, or interns to hospitals lead to one-to-many matchings, while the allocation of sets of optional courses to different groups of students leads to many-to-many matchings. Matching theory is particularly interested in the stability of matchings, in the attainability of stable matchings by means of appropriate institutional arrangements and rules, and in the potential room that such rules may leave for strategic behavior; see, for example, Roth and Sotomayor [128] for an introduction to the theory of matching. Discussing the preferences of the agents involved over sets of other agents is crucial in the analysis of any of those issues. Indeed, very different conclusions can be obtained depending on the assumptions that are made regarding the agents' preferences over sets of possible partners in matchings.

A third area where set rankings play an important role is at the boundary between economics and philosophy. Issues such as the examination of the instrumental and the intrinsic value of freedom of choice, or the possibility of formalizing notions such as equality of opportunity that can be turned into operational tools to discuss policy issues fall within that category. Several authors have raised issues of this type, have suggested different formulations to address them, and eventually advanced debates and proposals. To name but a few, the interested reader is referred to Arneson [4], Cohen [36], Dworkin [46, 47], Kolm [82, 83], Roemer [125, 126], and Sen [136, 138]. Once again, the very simple language of ranking sets has proven to be of value, for example, in supporting arguments and formalizing them at least partially.

Voting, matching, and some fundamental issues in ethics and economics are not the only sources of information for ranking sets. We shall mention a few more along the way in this survey. Our account concentrates on the issue of ranking sets, and we leave it to the interested reader to make further connections between the findings reported here and the broader areas of interest where these set rankings are of importance.

In order to provide the reader with a structured overview of and guide to the remainder of this chapter, we conclude this section by introducing a taxonomic structure of the problem of ranking sets.

We focus on set rankings that reflect interpretations in terms of some criterion or criteria of desirability or undesirability. By doing so, we exclude (at least) one interesting area where set rankings are of importance. Suppose we have a set of different statements, each of which has some degree of plausibility but all statements taken together may not be consistent. The problem of ranking sets of statements that are consistent is discussed in Rescher [123]. In that book, Rescher develops what he calls the theory of plausible reasoning. The basic idea underlying the analysis clearly is an instance of the problem of ranking sets of objects, and there have been some applications of the theory to problems in social choice; see, among others, Packard [109] and Packard and Heiner [110]. But the area is still relatively unexplored by social scientists and, in consideration of space limitations, we do not include it in this survey, even though we consider the issue important and interesting.

An important question to be answered before formulating desirable properties for rankings of sets is whether the elements of those sets are considered to be mutually exclusive options exactly one of which will finally materialize for the decision maker, or the members of the sets under consideration are to be 'consumed' or 'experienced' jointly by the agent. Although there are some contributions that analyze models where the objects in a set come jointly for an agent which we will discuss in more detail shortly, most of the standard approaches to the analysis of set rankings proceed by assuming that the alternatives are specified in a way such that only one of them will materialize in the end.

This leads us to a fundamental subdivision of the possible interpretations of set rankings discussed in this chapter. If only one of the options in a given set materializes for the decision maker, it has to be specified how this selection of a unique alternative comes about. The two natural contenders are the determination of the final outcome by a chance mechanism, and the choice made by the agent itself.

Figure 1 provides a diagrammatic illustration of the taxonomy used in this chapter. Based on this classification of approaches to the ranking of sets, the remainder of this survey is structured as follows.

In Section 3, we analyze the ranking of sets as a modelling tool for choice under complete uncertainty. In this framework, the elements of the sets to be ranked are interpreted as possible outcomes under

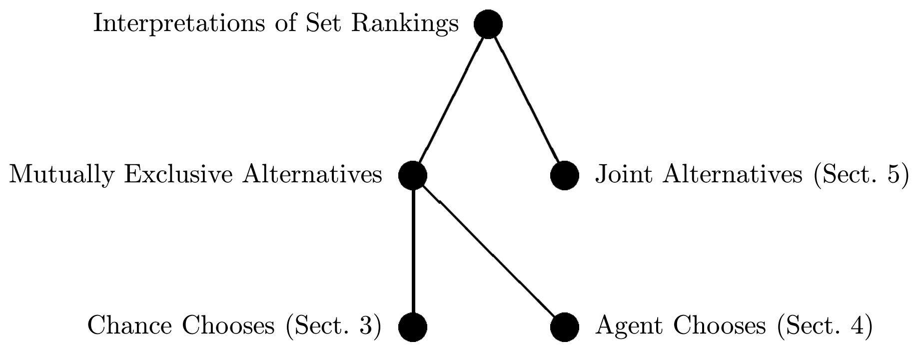

Figure 1 A Taxonomy of Set Rankings

uncertainty, and the final (unique) choice from a set of possible options is assumed to be made by means of some random device. The term 'complete uncertainty' refers to a situation where the agent knows the set of possible consequences of an action but cannot assign probabilities to those outcomes. One possible approach to address this issue proceeds by assuming that an agent faced with a decision problem of that nature first establishes a ranking of sets of possible outcomes and then chooses an action that leads to a best set of outcomes according to this ranking (provided, of course, that best sets exist).

Section 4 focuses on set rankings where it is up to the agent to choose a unique alternative from a set of feasible opportunities—an opportunity set. In addition to the indirect-utility criterion, according to which an agent ranks sets on the basis of their respective best elements with respect to an underlying ranking of alternatives, several alternative criteria have been proposed in the recent literature. In particular, rankings of opportunity sets on the basis of the freedom of choice they offer, the overall well-being they may induce for the agent, the flexibility they allow for, and the extent to which they represent uncertainty about consequences if those consequences differ from the objects of choice are discussed as possible criteria.

After discussing those two major branches of the literature on ranking sets in some detail, we return to the alternative interpretation mentioned earlier, where the sets to be ranked are not mutually exclusive. Section 5 deals with some examples of set rankings of that nature, where the elements of a set are assumed to materialize jointly. This situation can be observed in the analysis of matchings such as those encountered in college admission problems, where set rankings come into play because colleges need to rank sets of students based on their ranking of individual applicants. A related problem is the relationship between committee or assembly rankings and preferences over candidates for the committees in question. Unlike the rankings discussed earlier, some of these examples involve rankings of sets of a fixed cardinality.

Section 6 concludes the chapter.

3 COMPLETE UNCERTAINTY

3.1 Basic Concepts

Suppose that, in a given decision situation, an agent knows the set of possible outcomes associated with each feasible action, but the agent has no further information—in particular, there are no probabilities assigned to those outcomes which could be used to formulate a decision criterion. The approach to this decision problem under complete uncertainty reviewed in this section consists of ranking feasible actions on the basis of a ranking of the associated sets of possible outcomes. Given the underlying interpretation, it is natural to assume that this ranking of sets of possible outcomes extends the agent's ranking of the outcomes themselves in the sense that the relative ranking of singletons (which represent certain outcomes) according to the set ranking is the same as the relative ranking of the corresponding (single) elements in the respective sets according to the ranking of alternatives. This property motivates the use of the term extension rule for set rankings with this interpretation.

There are alternatives to the set-based model of choice under complete uncertainty described above. In many approaches followed in the literature (see, for example, Arrow and Hurwicz [7], Barrett and Pattanaik [16], Cohen and Jaffray [37], Luce and Raiffa [92], Maskin [95], and Milnor [97]), this situation is modelled according to the standard approach to decision theory by specifying a set of possible states of the world, and every pair of a feasible action and a state of the world leads to a specific outcome. Actions are then ranked on the basis of the vectors of contingent outcomes that they generate according to this approach. More recent work on modelling uncertainty has explored different variations that represent a departure from the classical decision-theoretic framework while still retaining a richer structure than the sets of consequences. See, for example, Bewley [22], Gilboa and Schmeidler [63], and Schmeidler [131].

It can be argued that the set-based approach surveyed here involves some loss of information as compared to the model based on vectors of contingent outcomes. In the model based on set rankings, it only matters whether or not a given outcome results in at least one state from choosing a given action. On the other hand, the alternative model allows the decision-maker to compare the sets of states that lead to given sets of outcomes. This includes comparisons based on the number of states leading to a given outcome. We may want to explicitly avoid attaching importance to the number of states where the same outcome arises, as this number may merely be the result of an arbitrary subdivision of otherwise indistinguishable states. But even then, other comparisons are possible, like those based on set containment. Hence, the purely set theoretic approach certainly entails some loss of information. Yet, there are circumstances where a ranking of sets can be considered the most appropriate way to model choice under complete uncertainty. For example, if the number of possible states of the world is large, an agent may find it too complex to take into account the entire vector of possible outcomes, and restricting attention to the set of possible outcomes may allow for a more tractable representation of the information that is available. Another scenario where the set-based model can be considered appropriate emerges in the context of a Rawlsian (Rawls [122]) veil-of-ignorance framework, where it may not be obvious how states and outcomes can be distinguished in a satisfactory fashion. See, for example, Bossert, Pattanaik, and Xu [32] and Pattanaik and Peleg [113] for further details and discussions.

We now turn to the formal framework employed in this section. Suppose there is a nonempty universal set of alternatives X with cardinality $|X|$, which may be finite or infinite (specific restrictions on the cardinality of X will be imposed for some of the results surveyed here). The set of all nonempty and finite subsets of X is denoted by $\mathcal{X}$. For $n \in \mathbb{N}$ with $n \leq |X|$, let $\mathcal{X}_n = \{A \in \mathcal{X} \mid |A| = n\}$, where $\mathbb{N}$ denotes the set of positive integers. In order to use a unified framework throughout this chapter, we focus on rankings of the sets in $\mathcal{X}$; we will, however, on occasion mention results obtained in models where X is infinite and $\mathcal{X}$ is replaced with the set of all nonempty subsets of X, including those with infinitely many elements. Furthermore, some models studied in the literature on set rankings include the empty set as an object to be ranked. Whenever we deviate from the standard model involving finite, nonempty sets only, we will clearly indicate that this is the case.

Let $R \subseteq X \times X$ be a binary relation on X which is interpreted as a preference relation of an agent, that is, $(x, y) \in R$ if and only if $x \in X$ is considered at least as good as $y \in X$ by the agent under consideration. For simplicity, we write xRy instead of $(x, y) \in R$. The strict preference relation P, the indifference relation I, and the noncomparability relation N corresponding to R are defined by letting, for all $x, y \in X$, xPy if and only if xRy and not yRx; xIy if and only if xRy and yRx; xNy if and only if not xRy and not yRx.

We use the following terminology for some standard properties of binary relations. R is reflexive if and only if xRx for all $x \in X$; R is complete if and only if xRy or yRx for all $x, y \in X$ such that $x \neq y$; R is transitive if and only if (xRy and yRz) implies xRz for all $x, y, z \in X$; R is quasi-transitive if and only if the corresponding P is transitive; R is antisymmetric if and only if xIy implies $x = y$ for all $x, y \in X$. A quasi-ordering is a reflexive and transitive relation, an ordering is a complete quasi-ordering, and a linear ordering is an antisymmetric ordering.

For most of the results discussed in this chapter, we assume that R is a fixed linear ordering. If X is finite, the linearity requirement can be considered a rather innocuous simplifying assumption, and a similar remark applies (to a slightly lesser extent) to situations where X is countably infinite. However, antisymmetry is a very stringent requirement when sets with a continuum of elements are to be compared; for that reason, we do not consider models where R is antisymmetric and X is required to be uncountable.

Let $A \in \mathcal{X}$. The set of best elements in A according to R is given by $\bar{\mathcal{B}}(A, R) = \{x \in A \mid xRy \text{ for all } y \in A\}$, and the set of worst elements in A according to R is $\bar{\mathcal{W}}(A, R) = \{x \in A \mid yRx \text{ for all } y \in A\}$. For simplicity, we omit the dependence of best and worst elements on R in this notation whenever R is fixed and this can be done without ambiguity. In those cases, we write the above sets as $\mathcal{B}(A)$ and $\mathcal{W}(A)$. If R is a fixed linear ordering, best and worst elements exist and are unique for all nonempty and finite sets $A \in \mathcal{X}$, and we denote them by $\max(A)$ and $\min(A)$.

In order to rank the elements of $\mathcal{X}$, we use a relation $\succeq \subseteq \mathcal{X} \times \mathcal{X}$ with strict preference relation $\succ$, indifference relation $\sim$, and noncomparability relation $\bowtie$. The interpretation of $\succeq$ is such that $A \succeq B$ if and only if the set of possible outcomes $A \in \mathcal{X}$ is considered at least as good as the set of possible outcomes $B \in \mathcal{X}$ by the decision maker.

Many of the axioms introduced in this chapter refer to the relationship between R and $\succeq$. However, for the sake of simplicity, we state them as properties of $\succeq$ alone. This can be done without any danger of ambiguity because R is assumed to be fixed.

To conclude this subsection, we introduce the extension-rule axiom. Given the interpretation assigned to $\succeq$ in this section, it is a very natural requirement. It will not always be imposed explicitly in this section because it often is implied by other axioms. Furthermore, it should be noted that its desirability stems from the interpretation given to $\succeq$ in this section; in some alternative models (for example, some of those discussed in Section 4), it will not be imposed because it lacks the intuitive plausibility it enjoys in the framework of the present section.

Extension rule requires the relative ranking of any two singleton sets according to $\succeq$ to be the same as the relative ranking of the corresponding alternatives themselves according to R.

Extension Rule: For all $x, y \in X$,

$$\{x\} \succeq \{y\} \Leftrightarrow xRy.$$

3.2 Best and Worst Elements

We begin our discussion of extension rules in the context of decision making under uncertainty by illustrating the fundamental role of best and worst elements in establishing a ranking of sets of alternatives. As an important step towards proving an impossibility theorem (which we will review in detail in Subsection 3.3), Kannai and Peleg [78] show that some plausible properties of an extension rule imply that any set $A \in \mathcal{X}$ must be indifferent to the set consisting of the best element and the worst element in A according to R. A related observation using variations of the axioms used by Kannai and Peleg appears in Bossert, Pattanaik, and Xu [32]. This latter result strengthens a similar observation due to Barberà, Barrett, and Pattanaik [11]; see also Arrow and Hurwicz [7] for this implication in a more traditional decision-theoretic context. The following axioms are used in those results.

Dominance: For all $A \in \mathcal{X}$, for all $x \in X$,

$$\begin{array}{ll} (i) & [xPy \text{ for all } y \in A] \Rightarrow A \cup \{x\} \succ A; \\ (ii) & [yPx \text{ for all } y \in A] \Rightarrow A \succ A \cup \{x\}. \end{array}$$

Simple Dominance: For all $x, y \in X$,

$$xPy \Rightarrow [\{x\} \succ \{x,y\} \text{ and } \{x,y\} \succ \{y\}].$$

Independence: For all $A, B \in \mathcal{X}$, for all $x \in X \setminus (A \cup B)$,

$$A \succ B \Rightarrow A \cup \{x\} \succeq B \cup \{x\}.$$

Extended Independence: For all $A, B \in \mathcal{X}$, for all $C \subseteq X \setminus (A \cup B)$,

$$A \succ B \Rightarrow A \cup C \succeq B \cup C.$$

Dominance is referred to as the Gärdenfors principle in Kannai and Peleg [78], in recognition of the use of this axiom in Gärdenfors [60]. See also Kim and Roush [80] for a discussion of the Gärdenfors axiom. This property requires that adding an element which is better (worse) than all elements in a given set A according to R leads to a set that is better (worse) than the original set according to $\succeq$. The dominance axiom appears to be a plausible requirement, given the interpretation of $\succeq$ analyzed in this section. Simple dominance is a weaker version of dominance that applies to the expansion of singleton sets by one element only; see, for example, Barberà [10]. As we will continue to do in the remainder of this chapter, we use the term 'simple' to indicate versions of axioms the scope of which is restricted to comparisons involving sets with at most two elements.

It is important to note that, if R is reflexive and antisymmetric and $\succeq$ is reflexive and quasi-transitive, simple dominance (and, thus, dominance) implies that $\succeq$ is an extension rule for R. This is why we do not always need to explicitly require the extension-rule axiom in several axiomatizations which actually lead to extensions.

Independence is the monotonicity axiom used in Kannai and Peleg [78]. It requires that if there exists a strict preference between two sets A and B, adding the same alternative to both sets does not reverse this strict ranking and, if $\succeq$ is not necessarily complete, neither does this augmentation of both sets turn the strict preference into a noncomparability. Again, this is a plausible axiom for a set ranking with an interpretation in terms of uncertain outcomes. A stronger version of independence requiring that strict preference is maintained in the consequent is discussed later—see Subsection 3.3. Extended independence is a strengthening of independence, introduced by Barberà, Barrett, and Pattanaik [11], which requires the same conclusion as independence if an arbitrary set (not necessarily a singleton) is added to both of two strictly ranked sets. See also Packard [109] for a similar condition. As shown in Theorem 3 below, independence and extended independence are equivalent in the presence of simple dominance, provided that $\succeq$ is a quasi-ordering.

Kannai and Peleg's [78] lemma states that if a reflexive and transitive relation $\succeq$ on $\mathcal{X}$ satisfies dominance and independence and R is a linear ordering, then any finite, nonempty subset A of the universal set X is indifferent to the set consisting of the best element and the worst element in A according to R. As noted in Bossert [24], transitivity of $\succeq$ can be weakened to quasi-transitivity in this result. Therefore, we obtain the following theorem.

Theorem 1 *Suppose R is a linear ordering on X and $\succeq$ is a reflexive and quasi-transitive relation on $\mathcal{X}$. If $\succeq$ satisfies dominance and independence, then $A \sim \{\max(A), \min(A)\}$ for all $A \in \mathcal{X}$.*

Proof. Let $A \in \mathcal{X}$. If A contains less than three elements, the claim follows immediately from the reflexivity of $\succeq$. Now suppose $|A| = n \geq 3$. Let $A = \{x_1, \ldots, x_n\}$ where, without loss of generality, $x_i P x_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. By repeated application of dominance and quasi-transitivity, it follows that $\{x_1\} \succ \{x_1, \ldots, x_{n-1}\}$, and independence implies

$$\{\max(A), \min(A)\} = \{x_1, x_n\} \succeq A. \tag{17.1}$$

Analogously, repeated application of dominance and quasi-transitivity yields $\{x_2,\ldots,x_n\} \succ \{x_n\}$ and, again using independence, we obtain

$$A \succeq \{x_1, x_n\} = \{\max(A), \min(A)\}. \tag{17.2}$$

Combining (17.1) and (17.2), we obtain $A \sim \{\max(A), \min(A)\}$. ■

An analogous theorem is obtained if dominance is weakened to simple dominance and $\succeq$ is assumed to be transitive rather than merely quasi-transitive. This theorem is proven in Bossert, Pattanaik, and Xu [32]; see also Barberà, Barrett, and Pattanaik [11] who state this result using extended independence.

Theorem 2 *Suppose R is a linear ordering on X and $\succeq$ is a quasi-ordering on $\mathcal{X}$. If $\succeq$ satisfies simple dominance and independence, then $A \sim \{\max(A), \min(A)\}$ for all $A \in \mathcal{X}$.*

Proof. We proceed by induction. Again, if A contains one or two elements only, $A \sim \{\max(A), \min(A)\}$ follows immedately by reflexivity. Now suppose $n > 2$ and

$$A \sim \{\max(A), \min(A)\} \text{ for all } A \in \mathcal{X} \text{ such that } |A| < n. \tag{17.3}$$

Let $A = \{x_1,\ldots,x_n\}$ be such that $x_i P x_{i+1}$ for all $i \in \{1,\ldots,n-1\}$. By simple dominance, $\{x_1\} \succ \{x_1, x_{n-1}\}$. By (17.3), $\{x_1, x_{n-1}\} \sim \{x_1,\ldots,x_{n-1}\}$ and, therefore, transitivity implies $\{x_1\} \succ \{x_1,\ldots,x_{n-1}\}$. Using independence, we obtain

$$\{x_1, x_n\} \succeq \{x_1,\ldots,x_n\}. \tag{17.4}$$

By (17.3), $\{x_2,\ldots,x_n\} \sim \{x_2, x_n\}$ and by simple dominance, $\{x_2, x_n\} \succ \{x_n\}$. Therefore, transitivity implies $\{x_2,\ldots,x_n\} \succ \{x_n\}$. By independence,

$$\{x_1,\ldots,x_n\} \succeq \{x_1, x_n\}. \tag{17.5}$$

(17.4) and (17.5) together imply $A \sim \{\max(A), \min(A)\}$, which completes the proof. ■

Nehring and Puppe [101] characterize all rankings that depend on best and worst elements only in a slightly different framework. They consider a model where the universal set X is endowed with some topological structure and analyze extensions of relations on this set to its entire power set, including uncountable subsets of X. In this setting, a continuity condition can be formulated, and it turns out that continuity and an independence condition also used in Puppe [119] together are necessary and sufficient for the relative ranking of any two sets to be the same as the relative ranking of the sets consisting of their respective best and worst elements only.

We conclude this subsection with a proof of the above-mentioned equivalence result concerning independence and extended independence. The method of proof is analogous to the one employed in the proof of Barberà, Barrett, and Pattanaik's [11] Theorem 3.4.

Theorem 3 *Suppose R is a linear ordering on X and $\succeq$ is a quasi-ordering on $\mathcal{X}$ satisfying simple dominance. $\succeq$ satisfies independence if and only if $\succeq$ satisfies extended independence.*

Proof. Clearly, extended independence implies independence. Now suppose $\succeq$ is a quasi-ordering on $\mathcal{X}$ satisfying simple dominance and independence. To prove that extended indepedence is satisfied, we first prove that we must have

$$\begin{aligned} &[\max(A) R \max(B) \text{ and } \min(A) R \min(B)] \\ &\Rightarrow \{\max(A), \min(A)\} \succeq \{\max(B), \min(B)\}. \end{aligned} \tag{17.6}$$

Suppose $\max(A) R \max(B)$ and $\min(A) R \min(B)$. If $\max(A) = \max(B)$,

$$\{\max(A), \min(A)\} \succeq \{\max(B), \min(A)\} \tag{17.7}$$

follows immediately from reflexivity. If $\max(A)P\max(B)$, simple dominance implies $\{\max(A)\} \succ \{\max(B)\}$, and independence implies (17.7). Analogously, it follows that $\{\max(B), \min(A)\} \succeq \{\max(B), \min(B)\}$. By transitivity, $\{\max(A), \min(A)\} \succeq \{\max(B), \min(B)\}$, which proves (17.6).

Now suppose $A \succ B$ and $C \subseteq X \setminus (A \cup B)$ for some $A, B, C \in \mathcal{X}$. By Theorem 2, it follows that

$$\{\max(A), \min(A)\} \succ \{\max(B), \min(B)\}. \tag{17.8}$$

In view of (17.6) and (17.8), $\max(B)R\max(A)$ and $\min(B)R\min(A)$ is impossible. Therefore, one of the following three cases must occur.

(i) $\max(A)R\max(B)$ and $\min(A)R\min(B)$;

(ii) $\max(A)P\max(B)$ and $\min(B)P\min(A)$;

(iii) $\max(B)P\max(A)$ and $\min(A)P\min(B)$.

In case (i), it follows that $\max(A\cup C)R\max(B\cup C)$ and $\min(A\cup C)R\min(B\cup C)$. By Theorem 2 and (17.6), we obtain $A \cup C \succeq B \cup C$.

In case (ii), it follows that

$$\max(A\cup C)R\max(B\cup C) \tag{17.9}$$

and

$$\min(B\cup C)R\min(A\cup C). \tag{17.10}$$

If $\min(B\cup C) = \min(A\cup C)$, (17.9), (17.6), and Theorem 2 imply $A \cup B \succeq B \cup C$.

Now suppose $\min(B\cup C)P\min(A\cup C)$. Because $\min(B)P\min(A)$ and $A\cap C = B\cap C = \emptyset$, it follows that $\min(A) = \min(A\cup C)$. Together with (17.8), this implies $\{\max(A), \min(A\cup C)\} \succ \{\max(B), \min(B)\}$. Clearly, $\min(B)R\min(B\cup C)$ and, together with (17.6) and transitivity, we obtain

$$\{\max(A), \min(A\cup C)\} \succ \{\max(B), \min(B\cup C)\}. \tag{17.11}$$

Again, it is clear that $\max(A\cup C)R\max(A)$. We can therefore distinguish the following two subcases.

(ii.a) $\max(A\cup C)P\max(A)$;

(ii.b) $\max(A\cup C) = \max(A)$.

Subcase (ii.a). By Theorem 2, we have

$$\{\max(A\cup C), \max(A), \min(A\cup C)\} \sim \{\max(A\cup C), \min(A\cup C)\} \tag{17.12}$$

and

$$\{\max(A\cup C), \max(B), \min(B\cup C)\} \sim \{\max(A\cup C), \min(B\cup C)\}. \tag{17.13}$$

By (17.11) and independence,

$$\{\max(A\cup C), \max(A), \min(A\cup C)\} \succeq \{\max(A\cup C), \max(B), \min(B\cup C)\}$$

and, using (17.12), (17.13), and transitivity, we obtain

$$\{\max(A\cup C), \min(A\cup C)\} \succeq \{\max(A\cup C), \min(B\cup C)\}.$$

By (17.9), (17.6), and transitivity, it follows that

$$\{\max(A\cup C), \min(A\cup C)\} \succeq \{\max(B\cup C), \min(B\cup C)\}$$

and, by Theorem 2, $A \cup C \succeq B \cup C$.

Subcase (ii.b). By (17.11), it follows that

$$\{\max(A \cup C), \min(A \cup C)\} \succ \{\max(B), \min(B \cup C)\}. \tag{17.14}$$

Because $A \cap C = \emptyset$, we must have $\max(A) P \max(C)$ and, hence, because we are in case (ii), $\max(A \cup C) P \max(B \cup C)$.

If $\max(B \cup C) = \max(B)$, $A \cup C \succeq B \cup C$ follows from (17.14) and Theorem 2.

Now suppose $\max(B \cup C) P \max(B)$. By (17.10) and transitivity, it follows that $\max(B \cup C) P \min(A \cup C)$. By Theorem 2,

$$\{\max(B \cup C), \max(A \cup C), \min(A \cup C)\} \sim \{\max(A \cup C), \min(A \cup C)\} \tag{17.15}$$

and

$$\{\max(B \cup C), \max(B), \min(B \cup C)\} \sim \{\max(B \cup C), \min(B \cup C)\}. \tag{17.16}$$

By (17.14) and independence,

$$\{\max(B \cup C), \max(A \cup C), \min(A \cup C)\} \succeq \{\max(B \cup C), \max(B), \min(B \cup C)\}$$

and, using (17.12), (17.13), and transitivity, we obtain

$$\{\max(A \cup C), \min(A \cup C)\} \succeq \{\max(B \cup C), \min(B \cup C)\}$$

and, by Theorem 2, $A \cup C \succeq B \cup C$.

Case (iii) is analogous to case (ii). ■

3.3 Impossibility Results

The axioms introduced in the previous subsection have very plausible intuitive interpretations. Surprisingly, it turns out that there are some incompatibilities between certain versions of the dominance and independence axioms. The first impossibility theorem in this framework is proven in Kannai and Peleg's [78] seminal paper on the subject. They show that if R is a linear ordering and X contains at least six elements, there exists no ordering $\succeq$ on $\mathcal{X}$ satisfying dominance and independence.

Theorem 4 *Suppose $|X| \geq 6$ and R is a linear ordering on X. There exists no ordering $\succeq$ on $\mathcal{X}$ satisfying dominance and independence.*

Proof. Let $|X| \geq 6$, and suppose R is a linear ordering on X. By way of contradiction, suppose $\succeq$ is an ordering on $\mathcal{X}$ satisfying dominance and independence. Consider $x_1, \ldots, x_6 \in X$ such that $x_i P x_{i+1}$ for all $i \in \{1, \ldots, 5\}$. Suppose first that

$$\{x_3\} \succ \{x_2, x_5\}. \tag{17.17}$$

In that case, independence implies $\{x_3, x_6\} \succeq \{x_2, x_5, x_6\}$ and, by Theorem 1 and the transitivity of $\succeq$, $\{x_3, x_4, x_5, x_6\} \succeq \{x_2, x_3, x_4, x_5, x_6\}$, which contradicts dominance. Therefore, (17.17) must be false and, because $\succeq$ is complete, it follows that $\{x_2, x_5\} \succeq \{x_3\}$. By dominance and transitivity, $\{x_2, x_5\} \succ \{x_4\}$. Using independence again, it follows that $\{x_1, x_2, x_5\} \succeq \{x_1, x_4\}$. By Theorem 1 and transitivity, $\{x_1, x_2, x_3, x_4, x_5\} \succeq \{x_1, x_2, x_3, x_4\}$. Because this represents a violation of dominance, we obtain the desired contradiction. ■

As noted by Kannai and Peleg [78] (who credit Peter Fishburn with this observation), dominance-type axioms tend to rule out rankings of sets that are based on 'total-goodness' criteria with respect to R, and independence axioms of the nature introduced above are designed to prevent set rankings to be based on 'average-goodness' considerations. The above theorem shows that not only those two types of extension rules are inadmissible—there cannot exist any ordering on $\mathcal{X}$ satisfying dominance and independence.

The mimimal number of elements in X cannot be reduced in the above theorem: if $|X| \leq 5$, orderings on $\mathcal{X}$ satisfying dominance and independence do exist; see, for example, Bandyopadhyay [8].

Barberà and Pattanaik [14] provide a similar impossibility result. If independence is strengthened by requiring the addition of an alternative to each of two strictly ranked sets to preserve this strict ranking, an impossibility result obtains even if dominance is weakened to simple dominance and no further properties (such as completeness or transitivity) are imposed on $\succeq$. The above-described strengthening of independence is defined as follows.

Strict Independence: For all $A, B \in \mathcal{X}$, for all $x \in X \setminus (A \cup B)$,

$$A \succ B \Rightarrow A \cup \{x\} \succ B \cup \{x\}.$$

We obtain

Theorem 5 *Suppose $|X| \geq 3$ and R is a linear ordering on X. There exists no binary relation $\succeq$ on $\mathcal{X}$ satisfying simple dominance and strict independence.*

Proof. Suppose, by way of contradiction, that $|X| \geq 3$, R is a linear ordering on X, and $\succeq$ is a binary relation on $\mathcal{X}$ satisfying simple dominance and strict independence. Let $x_1, x_2, x_3 \in X$ be such that $x_1 P x_2 P x_3$. By simple dominance, $\{x_1\} \succ \{x_1, x_2\}$, and strict independence implies

$$\{x_1, x_3\} \succ \{x_1, x_2, x_3\}. \tag{17.18}$$

Simple dominance also implies $\{x_2, x_3\} \succ \{x_3\}$ and, using strict independence again, it follows that $\{x_1, x_2, x_3\} \succ \{x_1, x_3\}$, contradicting (17.18). ■

As is easy to see, the minimal number of elements in X required for the above theorem cannot be reduced.

We conclude this section with an impossibility theorem that is similar to the one proven by Kannai and Peleg. If a neutrality axiom is added to dominance and independence, the minimal number of elements in X can be reduced to four, and the impossibility result remains true. This observation is a consequence of a strengthening of Lemma 2 in Bossert [24]. As a preliminary step in the proof of a characterization result (see the following subsection for details), Bossert [24] shows that there must be some sets that are noncomparable according to a transitive relation $\succeq$ that satisfies dominance, independence, and the neutrality axiom mentioned above if X contains at least five elements. As shown in Theorem 6 below, this result is true for sets with four elements as well.

Neutrality requires that the labelling of the alternatives is irrelevant in establishing the ranking $\succeq$. This axiom, which also appears in Nitzan and Pattanaik [103] and Pattanaik and Peleg [113], is defined as follows.

Neutrality: For all $A, B \in \mathcal{X}$, for all one-to-one mappings $\varphi: A \cup B \to X$,

$$\begin{aligned}&([xRy \Leftrightarrow \varphi(x)R\varphi(y) \text{ and } yRx \Leftrightarrow \varphi(y)R\varphi(x)] \text{ for all } x \in A, \text{ for all } y \in B)\\ &\quad \Rightarrow (A \succeq B \Leftrightarrow \varphi(A) \succeq \varphi(B) \text{ and } B \succeq A \Leftrightarrow \varphi(B) \succeq \varphi(A)).\end{aligned}$$

We now obtain

Theorem 6 *Suppose $|X| \geq 4$ and R is a linear ordering on X. There exists no ordering $\succeq$ on $\mathcal{X}$ satisfying dominance, independence, and neutrality.*

Proof. Suppose, by way of contradiction, that $|X| \geq 4$, R is a linear ordering on X, and $\succeq$ is an ordering on $\mathcal{X}$ satisfying dominance, independence, and neutrality. Let $x_1, x_2, x_3, x_4 \in X$ be such that $x_1 P x_2 P x_3 P x_4$. Suppose

$$\{x_1, x_4\} \succeq \{x_2\}. \tag{17.19}$$

By dominance and transitivity, it follows that $\{x_1, x_4\} \succ \{x_3\}$. By neutrality, $\{x_2, x_4\} \succ \{x_3\}$, and independence implies $\{x_1, x_2, x_4\} \succeq \{x_1, x_3\}$. By Theorem 1, $\{x_1, x_2, x_3, x_4\} \succeq \{x_1, x_2, x_3\}$, contradicting dominance. Therefore, (17.19) cannot be true, and the completeness of $\succeq$ implies $\{x_2\} \succ \{x_1, x_4\}$. By neutrality, $\{x_2\} \succ \{x_1, x_3\}$, and independence implies $\{x_2, x_4\} \succeq \{x_1, x_3, x_4\}$. By Theorem 1, $\{x_2, x_3, x_4\} \succeq \{x_1, x_2, x_3, x_4\}$, and we obtain a contradiction to dominance which completes the proof. ■

The conclusion of Theorem 6 is not true if X contains less than four elements. For example, suppose $X = \{x_1, x_2, x_3\}$ and $x_1 P x_2 P x_3$. An ordering $\succeq$ satisfying dominance, independence, and neutrality is given by

$$\{x_1\} \succ \{x_1, x_2\} \succ \{x_1, x_3\} \sim \{x_2\} \sim \{x_1, x_2, x_3\} \succ \{x_2, x_3\} \succ \{x_3\}.$$

3.4 Characterizations

Possibility results can be obtained by modifying Kannai and Peleg's axioms. For example, Barberà and Pattanaik [14] show that there exist orderings on $\mathcal{X}$ satisfying (extended) independence and an axiom that is intermediate in strength between dominance and simple dominance. Furthermore, Fishburn [51] (see also Holzman [74]) provides an example of an extension rule satisfying dominance and another variant of the independence condition.

In this subsection, we review some characterization results that provide, in addition to the existence results they imply, the set of all rankings satisfying the axioms in question. In several of these axiomatizations, best and worst elements play a crucial role, due to the observation that conditions closely related to those introduced in Subsection 3.2 are employed.

The first characterization result we present is due to Barberà, Barrett, and Pattanaik [11], who state a version using extended independence instead of independence.

As illustrated in Theorem 2, if simple dominance and independence are satisfied, any set $A \in \mathcal{X}$ must be indifferent to the set consisting of the best and the worst element of A only. As a consequence, knowledge of the restriction of the ranking to singletons and two-element sets is sufficient to recover the entire ordering $\succeq$. Moreover, to ensure that simple dominance and independence are satisfied by the induced ordering on $\mathcal{X}$, a given ranking of singletons and two-element sets must satisfy certain properties. In order to state the resulting characterization theorem formally, some more notation is required. Let $\mathcal{X}_{1,2} = \{A \in \mathcal{X} \mid |A| \leq 2\}$, that is, $\mathcal{X}_{1,2}$ is the set of all subsets of X of cardinality one or two. Let $\succeq_{1,2}$ be an ordering on $\mathcal{X}_{1,2}$. The following version of the independence axiom, restricted to comparisons involving elements of $\mathcal{X}_{1,2}$, is required for the statement of our next theorem.

Restricted Independence: For all $x, y, z, w, v \in X$ such that xRy and zRw,

$$\begin{array}{ll} (i) & [\{x,y\} \succ \{z,w\} \text{ and } x \neq v \text{ and } vPz] \Rightarrow \{\max(\{v,x\}), y\} \succeq \{v,w\}; \\ (ii) & [\{x,y\} \succ \{z,w\} \text{ and } w \neq v \text{ and } yPv] \Rightarrow \{v,x\} \succeq \{z, \min(\{v,w\})\}. \end{array}$$

We say that $\succeq$ is a maxmin-based ordering if and only if there exists an ordering $\succeq_{1,2}$ on $\mathcal{X}_{1,2}$ satisfying simple dominance and restricted independence such that, for all $A, B \in \mathcal{X}$,

$$A \succeq B \Leftrightarrow \{\max(A), \min(A)\} \succeq_{1,2} \{\max(B), \min(B)\}.$$

We now obtain

Theorem 7 *Suppose R is a linear ordering on X and $\succeq$ is an ordering on $\mathcal{X}$. $\succeq$ satisfies simple dominance and independence if and only if $\succeq$ is a maxmin-based ordering.*

Proof. By Theorem 3.4 of Barberà, Barrett, and Pattanaik [11], the maxmin-based orderings are characterized by simple dominance and extended independence, given that $\succeq$ is an ordering. The result now follows from Theorem 3. ■

By adding some axioms to simple dominance and independence, Bossert, Pattanaik, and Xu [32] characterize special cases of the maxmin-based orderings. In particular, two orderings that treat best and worst elements in a lexicographic fashion are considered. The minmax ordering $\succeq_{mnx}$ and the maxmin ordering $\succeq_{mxn}$ are defined by letting, for all $A, B \in \mathcal{X}$,

$$\begin{aligned} A \succeq_{mnx} B \quad \Leftrightarrow \quad & (\min(A) P \min(B) \text{ or} \\ & [\min(A) = \min(B) \text{ and } \max(A) R \max(B)]) \end{aligned}$$

and

$$\begin{aligned} A \succeq_{mxn} B \quad \Leftrightarrow \quad & (\max(A) P \max(B) \text{ or} \\ & [\max(A) = \max(B) \text{ and } \min(A) R \min(B)]), \end{aligned}$$

respectively.

The minmax ordering uses the worst element as the primary criterion for ranking sets of uncertain outcomes, and the best element merely plays the role of a tie-breaker. The converse is true for the maxmin ordering. One possible interpretation of those rules is that minmax represents 'uncertainty-averse' behaviour, whereas maxmin is an 'uncertainty-seeking' ranking. Indeed, the following definitions of uncertainty aversion and uncertainty appeal can be used to obtain characterizations of those rules.

Simple Uncertainty Aversion: For all $x, y, z \in X$,

$$xPyPz \Rightarrow \{y\} \succ \{x, z\}.$$

Simple Uncertainty Appeal: For all $x, y, z \in X$,

$$xPyPz \Rightarrow \{x, z\} \succ \{y\}.$$

It is clear how those axioms relate to the attitudes towards uncertainty mentioned above. Simple uncertainty aversion postulates that receiving a certain outcome y is always preferred to the possibility of receiving either a better outcome or a worse outcome than y. In contrast, the possibility of getting a better outcome than y, even though this is associated with the possibility of a worse outcome than y, is always preferred to receiving y with certainty if $\succeq$ represents uncertainty appeal. These axioms are weakenings of Bossert's [27] uncertainty-aversion and uncertainty-appeal axioms; see Bossert, Pattanaik, and Xu [32] for further discussion.

Given simple dominance and independence, the only axioms that have to be added to these conditions representing specific attitudes towards uncertainty are the following monotonicity properties.

Simple Top Monotonicity: For all $x, y, z \in X$,

$$xPyPz \Rightarrow \{x, z\} \succ \{y, z\}.$$

Simple Bottom Monotonicity: For all $x, y, z \in X$,

$$xPyPz \Rightarrow \{x, y\} \succ \{x, z\}.$$

The intuition underlying simple top monotonicity is straightforward. Given a singleton $\{z\}$ and two distinct alternatives x and y both of which are better than $\{z\}$, adding the better of x and y to the singleton set is better than adding the worse of those two alternatives. Simple bottom monotonicity is the dual of this axiom that applies to the addition of worse alternatives to a singleton set.

We obtain

Theorem 8 *Suppose R is a linear ordering on X and $\succeq$ is an ordering on $\mathcal{X}$. $\succeq$ satisfies simple dominance, independence, simple uncertainty aversion, and simple top monotonicity if and only if $\succeq = \succeq_{mnx}$.*

Proof. See Bossert, Pattanaik, and Xu [32]. ■

Theorem 9 *Suppose R is a linear ordering on X and $\succeq$ is an ordering on $\mathcal{X}$. $\succeq$ satisfies simple dominance, independence, simple uncertainty appeal, and simple bottom monotonicity if and only if $\succeq = \succeq_{mxn}$.*

Proof. See Bossert, Pattanaik, and Xu [32]. ■

As an alternative to the maxmin-based orderings, Pattanaik and Peleg [113] consider extension rules that put more emphasis on best elements or on worst elements rather than treating best and worst elements symmetrically. In particular, they provide characterizations of the leximin and the leximax orderings on $\mathcal{X}$ (see also Holzman [75]). The leximin ordering proceeds by first considering the worst elements in two sets A and B to be compared. If the worst element in one of the sets is better than the worst element in the other according to R, the former set is declared better than the latter. If the two worst elements are indifferent according to R (if R is linear, this means that the two sets have the same worst element), we eliminate the worst elements from both sets and consider the remaining sets. Again, if the worst elements of the reduced sets are identical, they are removed from both sets, the reduced sets are considered, and the procedure continues in that fashion. If this successive elimination of identical worst elements leads to a situation where one of the original sets is reduced to the empty set but the reduced set of the second is nonempty, the latter set is declared better than the former. Indifference is obtained only if the sets are identical. Clearly, this procedure is well-defined because we only consider finite subsets of X. The leximax ordering is dual to the leximin ordering in the sense that we start with the best elements and successively work our way down through worse and worse elements, and if one set is reduced to the empty set but the other is not, the former is declared better than the latter.

To define these rules formally, let R be a linear ordering on X, and consider the following rank-ordered permutations of the index sets represeting the numbering of elements in a set. For $A = \{a_1, \ldots, a_{|A|}\} \in \mathcal{X}$, let $\sigma_A \colon \{1, \ldots, |A|\} \to \{1, \ldots, |A|\}$ be such that $a_{\sigma_A(i+1)} R a_{\sigma_A(i)}$ for all $i \in \{1, \ldots, |A| - 1\}$, and let $\rho_A \colon \{1, \ldots, |A|\} \to \{1, \ldots, |A|\}$ be such that $a_{\rho_A(i)} R a_{\rho_A(i+1)}$ for all $i \in \{1, \ldots, |A| - 1\}$. The leximin ordering $\succeq^L_{\min}$ on $\mathcal{X}$ is defined by letting, for all $A, B \in \mathcal{X}$,

$$\begin{aligned} A \succeq^L_{\min} B \quad \Leftrightarrow \quad & (A = B \text{ or} \\ & [|A| > |B| \text{ and } a_{\sigma_A(i)} = b_{\sigma_B(i)} \text{ for all } i \in \{1, \ldots, |B|\}] \text{ or} \\ & [\exists i \in \{1, \ldots, \min\{|A|, |B|\}\} \text{ such that} \\ & a_{\sigma_A(j)} = b_{\sigma_B(j)} \text{ for all } j < i \text{ and } a_{\sigma_A(i)} P b_{\sigma_B(i)}]). \end{aligned}$$

Analogously, the leximax ordering $\succeq^L_{\max}$ is defined by letting, for all $A, B \in \mathcal{X}$,

$$\begin{aligned} A \succeq^L_{\max} B \quad \Leftrightarrow \quad & (A = B \text{ or} \\ & [|A| < |B| \text{ and } a_{\rho_A(i)} = b_{\rho_B(i)} \text{ for all } i \in \{1, \ldots, |A|\}] \text{ or} \\ & [\exists i \in \{1, \ldots, \min\{|A|, |B|\}\} \text{ such that} \\ & a_{\rho_A(j)} = b_{\rho_B(j)} \text{ for all } j < i \text{ and } a_{\rho_A(i)} P b_{\rho_B(i)}]). \end{aligned}$$

Pattanaik and Peleg's [113] characterizations of the leximin and the leximax orderings are obtained by adding specific independence conditions to dominance and neutrality.

Bottom Independence: For all $A, B \in \mathcal{X}$, for all $x \in X$ such that yPx for all $y \in A \cup B$,

$$A \succ B \Rightarrow A \cup \{x\} \succ B \cup \{x\}.$$

Top Independence: For all $A, B \in \mathcal{X}$, for all $x \in X$ such that xPy for all $y \in A \cup B$,

$$A \succ B \Rightarrow A \cup \{x\} \succ B \cup \{x\}.$$

Disjoint Independence: For all $A, B \in \mathcal{X}$ such that $A \cap B = \emptyset$, for all $x \in X \setminus (A \cup B)$,

$$A \succ B \Leftrightarrow A \cup \{x\} \succ B \cup \{x\}.$$

Bottom independence is a weakening of strict independence that only applies to situations where the alternative to be added to each of two sets is worse than any alternative in either of the two original sets. Top independence is its dual, where the added alternative is better than all of those present in the original sets. Disjoint independence is related to an axiom used by Fishburn [51]. Again, the scope of the independence condition is limited, this time to situations where the original sets to each of which an alternative is to be added are disjoint.

The following two theorems are due to Pattanaik and Peleg [113] who also prove generalized versions where R is not assumed to be linear; see also Holzman [75].

Theorem 10 *Suppose $|X| \geq 4$, R is a linear ordering on X, and $\succeq$ is a quasi-ordering on $\mathcal{X}$. $\succeq$ satisfies dominance, neutrality, bottom independence, and disjoint independence if and only if $\succeq = \succeq^L_{\min}$.*

Proof. See Pattanaik and Peleg [113]. ■

Theorem 11 *Suppose $|X| \geq 4$, R is a linear ordering on X, and $\succeq$ is a quasi-ordering on $\mathcal{X}$. $\succeq$ satisfies dominance, neutrality, top independence, and disjoint independence if and only if $\succeq = \succeq^L_{\max}$.*

Proof. See Pattanaik and Peleg [113]. ■

Note that, in the above theorems, $\succeq$ does not have to be assumed to be complete—this property follows as a consequence of the remaining axioms. See Remark 5.1 of Pattanaik and Peleg [113]. This observation suggests an alternative way of resolving Kannai and Peleg's [78] impossibility theorem. Instead of weakening one of their axioms in order to obtain possibilities, the requirement that $\succeq$ is an ordering could be weakened. This is the approach followed in Bossert [24], where $\succeq$ is merely assumed to be a quasi-ordering. As noted in Remark 2 of Kannai and Peleg [78], the minmax dominance quasi-ordering defined below satisfies dominance and independence. Bossert [24] shows that this quasi-ordering can be characterized if neutrality is added as a requirement. The minmax dominance quasi-ordering $\succeq_D$ is defined by letting, for all $A, B \in \mathcal{X}$,

$$A \succeq_D B \Leftrightarrow [\max(A) R \max(B) \text{ and } \min(A) R \min(B)].$$

Bossert's [24] characterization result is stated for universal sets X with at least five elements but, as shown below, it is also true for $|X| = 4$. Therefore, we obtain

Theorem 12 *Suppose $|X| \geq 4$, R is a linear ordering on X, and $\succeq$ is a quasi-ordering on $\mathcal{X}$. $\succeq$ satisfies simple dominance, independence, and neutrality if and only if $\succeq = \succeq_D$.*

Proof. Clearly, $\succeq_D$ is a quasi-ordering satisfying the required axioms. Conversely, suppose $\succeq$ is a quasi-ordering that satisfies dominance, independence, and neutrality. It is sufficient to show that, for all $A, B \in \mathcal{X}$,

$$A \sim_D B \quad \Rightarrow \quad A \sim B; \tag{17.20}$$
$$A \succ_D B \quad \Rightarrow \quad A \succ B; \tag{17.21}$$
$$A \bowtie_D B \quad \Rightarrow \quad A \bowtie B. \tag{17.22}$$

To prove (17.20), note that, by definition, $A \sim_D B$ is equivalent to $\max(A) = \max(B)$ and $\min(A) = \min(B)$ which, by Theorem 1, implies $A \sim B$.

Now suppose $A \succ_D B$. By definition of $\succeq_D$, at least one of the following two cases must occur.

(i) $\max(A) P \max(B)$ and $\min(A) R \min(B)$;

(ii) $\max(A) R \max(B)$ and $\min(A) P \min(B)$.

In case (i), dominance implies $\{\max(A), \max(B), \min(B)\} \succ \{\max(B), \min(B)\}$. By Theorem 1, we obtain

$$\{\max(A), \min(B)\} \succ \{\max(B), \min(B)\}. \tag{17.23}$$

If $\min(A) = \min(B)$, reflexivity implies

$$\{\max(A), \min(A)\} \succeq \{\max(A), \min(B)\}, \tag{17.24}$$

and if $\min(A) P \min(B)$, (17.24) follows from dominance and independence. Combining (17.23) and (17.24), transitivity implies, using Theorem 1, $A \succ B$. Case (ii) is analogous, and (17.21) is proven.

Finally, to prove (17.22), suppose $A \bowtie_D B$. Again, according to the definition of $\succeq_D$, there are two possible cases.

(i) $\max(A) P \max(B)$ and $\min(B) P \min(A)$;

(ii) $\max(B) P \max(A)$ and $\min(A) P \min(B)$.

Consider case (i). Suppose first that $|B| = 1$. By the argument used in Theorem 6, it follows that, for all $x, y, z \in X$ such that $xPyPz$,

$$\{y\} \bowtie \{x, z\} \tag{17.25}$$

and, therefore, $\{\max(B)\} = \{\min(B)\} \bowtie \{\max(A), \min(A)\}$ which, by Theorem 1, implies $A \bowtie B$.

Now suppose $|B| \geq 2$. If $\{\max(A), \min(A)\} \succeq \{\max(B), \min(B)\}$, transitivity and dominance imply $\{\max(A), \min(A)\} \succ \{\min(B)\}$, contradicting (17.25) and, analogously, if $\{\max(B), \min(B)\} \succeq \{\max(A), \min(A)\}$, transitivity and dominance yield $\{\max(B)\} \succ \{\max(A), \min(A)\}$, again contradicting (17.25). Therefore, we must have $\{\max(A), \min(A)\} \bowtie \{\max(B), \min(B)\}$ and hence, by Theorem 1 again, $A \bowtie B$.

Case (ii) is analogous. ■

Bossert, Pattanaik, and Xu [32] provide refinements of the maxmin and minmax rules by considering their lexicographic extensions. According to the lexicographic minmax rule, if there is a strict preference between two sets according to the minmax rule, then this strict preference is respected by the lexicographic minmax rule. However, if the two sets are indifferent according to minmax, they are not necessarily declared indifferent according to the lexicographic minmax ordering but, instead, the best and worst elements are removed from both sets, and the reduced sets are again compared according to minmax, and so on. If this procedure yields a situation where one of the reduced sets is empty but the other is not, then the latter is declared better than the former. The lexicographic maxmin ordering is defined analogously. See Bossert, Pattanaik, and Xu [32] for further details and characterizations of those orderings.

Another characterization of a lexicographic-type ordering can be found in Heiner and Packard [71]. Because this result is obtained in a slightly different framework than the one we focus on here (R is not assumed to be linear, and the empty set is included as one of the sets to be ranked in Heiner and Packard's paper), we do not provide further details and refer the reader to the original paper instead.

Finally, Nehring and Puppe [101] provide several characterizations of rules that are based on best and worst elements in a continuous setting. Again, because they work with a different model where the set X has some topological structure, we do not include formal statements of those results in this survey; see Nehring and Puppe [101] for details.

The last result we want to discuss in some detail in this section differs from those analyzed so far in that it provides an alternative class of extension rules that are based on median elements rather than best or worst alternatives. The characterization of those median-based rules is due to Nitzan and Pattanik [103].

For any $A = \{a_1, \ldots, a_{|A|}\} \in \mathcal{X}$ such that, without loss of generality, $a_i P a_{i+1}$ for all $i \in \{1, \ldots, |A|-1\}$, let $\mathrm{med}(A) = \{a_{(|A|+1)/2}\}$ if $|A|$ is odd and $\mathrm{med}(A) = \{a_{|A|/2}, a_{|A|/2+1}\}$ if $|A|$ is even. The following axioms are used in Nitzan and Pattanaik's [103] characterization.

Intermediate Independence: For all $A, B \in \mathcal{X}$, for all $x, y \in X$ such that $xPzPy$ for all $z \in A \cup B$,

$$A \succeq B \Rightarrow A \cup \{x, y\} \succeq B \cup \{x, y\}.$$

Even-Numbered Extension of Equivalence: For all $A \in \mathcal{X}$ such that $|A|$ is even, for all $x, y \in X \setminus A$,

$$[A \cup \{x\} \sim \{x\} \text{ and } A \cup \{y\} \sim \{y\}] \Rightarrow A \cup \{x, y\} \sim \{x, y\}.$$

Odd-Numbered Neutrality: For all $A, B \in \mathcal{X}$ such that $|A|$ and $|B|$ are odd, for all one-to-one mappings $\varphi\colon A \cup B \to X$,

$$\begin{aligned}
&([xRy \Leftrightarrow \varphi(x)R\varphi(y) \text{ and } yRx \Leftrightarrow \varphi(y)R\varphi(x)] \text{ for all } x \in A, \text{ for all } y \in B) \\
&\quad \Rightarrow (A \succeq B \Leftrightarrow \varphi(A) \succeq \varphi(B) \text{ and } B \succeq A \Leftrightarrow \varphi(B) \succeq \varphi(A)).
\end{aligned}$$

Odd-Numbered Duality: For all $A, B \in \mathcal{X}$ such that $|A|$ and $|B|$ are odd, for all one-to-one mappings $\varphi\colon A \cup B \to X \setminus (A \cup B)$,

$$\begin{aligned}
&([xRy \Leftrightarrow \varphi(y)R\varphi(x) \text{ and } yRx \Leftrightarrow \varphi(x)R\varphi(y)] \text{ for all } x \in A, \text{ for all } y \in B) \\
&\quad \Rightarrow (A \succeq B \Leftrightarrow \varphi(B) \succeq \varphi(A) \text{ and } B \succeq A \Leftrightarrow \varphi(A) \succeq \varphi(B)).
\end{aligned}$$

Intermediate independence is a variation of extended independence, where the scope of the axiom is limited to additions of two-element sets, one element of which is better than everything in the original sets and one element of which is worse. Even-numbered extension of equivalence states that whenever adding a set A to a singleton $\{x\}$ is a matter of indifference and adding the same set A to a singleton $\{y\}$ is a matter of indifference as well, then adding A to $\{x, y\}$ must be a matter of indifference, too. Again, the scope of the axiom is restricted: it only applies to situations where the set to be added has an even number of elements. A stronger version of this axiom is used in Packard [108]. Odd-numbered neutrality weakens neutrality by restricting its scope to sets with an odd number of elements. Finally, odd-numbered duality can be interpreted as a weakening of a 'mirror-image' variation of odd-numbered neutrality. Note that the range of the function φ in the definition of odd-numbered duality is $X \setminus (A \cup B)$ rather than X, which is why this axiom is weaker than the straightforward analogue of odd-numbered neutrality. See Nitzan and Pattanaik [103] for further remarks concerning the interpretation of these axioms.

The following theorem characterizes a class of median-based orderings.

Theorem 13 *Suppose $|X| \geq 6$, R is a linear ordering on X, and $\succeq$ is an ordering on $\mathcal{X}$ satisfying extension rule. $\succeq$ satisfies intermediate independence, even-numbered extension of equivalence, odd-numbered neutrality, and odd-numbered duality if and only if $A \sim med(A)$ for all $A \in \mathcal{X}$.*

Proof. It is straightforward to verify that, given the extension-rule axiom, the median-based orderings satisfy the required axioms. Now suppose $\succeq$ satisfies intermediate independence, even-numbered extension of equivalence, odd-numbered neutrality, and odd-numbered duality. Let $A \in \mathcal{X}$. If $|A| \leq 2$, $A \sim \text{med}(A)$ follows from reflexivity. Because X contains at least six elements and R is linear, there exist $x_1, \ldots, x_6 \in X$ such that $x_i P x_{i+1}$ for all $i \in \{1, \ldots, 5\}$. Because $\succeq$ is complete, one of the following cases must occur.

(i) $\{x_1, x_2, x_3\} \succ \{x_2\}$;

(ii) $\{x_2\} \succ \{x_1, x_2, x_3\}$;

(iii) $\{x_1, x_2, x_3\} \sim \{x_2\}$.

In case (i), odd-numbered neutrality implies $\{x_4, x_5, x_6\} \succ \{x_5\}$, and odd-numbered duality implies $\{x_5\} \succ \{x_4, x_5, x_6\}$, a contradiction. Case (ii) leads to an analogous contradiction and, therefore, case (iii) must apply. By odd-numbered neutrality,

$$A \sim \text{med}(A) \quad \text{for all } A \in \mathcal{X} \text{ such that } |A| = 3. \tag{17.26}$$

Now consider $A \in \mathcal{X}$ such that $|A|$ is odd and $|A| \geq 5$. Without loss of generality, let $|A| = 2n + 1$ for some $n \neq 2$ and $A = \{x_1, \ldots, x_{2n+1}\}$ with $x_i P x_{i+1}$ for all $i \in \{1, \ldots, 2n\}$. By (17.26), $\{x_{n+1}\} \sim \{x_n, x_{n+1}, x_{n+2}\}$, and intermediate independence implies

$$\{x_{n-1}, x_{n+1}, x_{n+3}\} \sim \{x_{n-1}, x_n, x_{n+1}, x_{n+2}, x_{n+3}\}.$$

Using (17.26) again, it follows that $\{x_{n+1}\} \sim \{x_{n-1}, x_{n+1}, x_{n+3}\}$, and transitivity implies $\{x_{n+1}\} \sim \{x_{n-1}, x_n, x_{n+1}, x_{n+2}, x_{n+3}\}$. Because A is finite, this procedure can be applied repeatedly to conclude that $A \sim \text{med}(A)$.

It remains to be shown that A is indifferent to med(A) for all A with an even number of elements greater than two. Let, without loss of generality, $A = \{x_1, \ldots, x_{2n}\}$ with $n \geq 2$ and $x_i P x_{i+1}$ for all $i \in \{1, \ldots, 2n - 1\}$. Because $A \sim \text{med}(A)$ for all A with an odd number of elements, it follows that

$$\{x_1, \ldots, x_{n-1}, x_n, x_{n+2}, \ldots, x_{2n}\} \sim \{x_n\}$$

and

$$\{x_1, \ldots, x_{n-1}, x_{n+1}, x_{n+2}, \ldots, x_{2n}\} \sim \{x_{n+1}\}.$$

By even-numbered extension of equivalence, it follows that $A \sim \{x_n, x_{n+1}\} = \text{med}(A)$. ■

Note that, unlike other characterizations in this section, the above result requires explicit use of the extension-rule axiom. This is the case because, without requiring the axiom, median-based rules that are not extension rules would be included in the characterization; for example, the ordering that declares all sets pairwise indifferent is a median-based ordering satisfying all other axioms in the above theorem statement.

4 OPPORTUNITIES

4.1 Indirect Utility

Much of the formal framework introduced in the context of choice under complete uncertainty can be imported into this section with some changes in interpretation. As before, we assume that there exists a nonempty set of alternatives X and an individual preference ordering R on X which, unless stated otherwise, is assumed to be linear. Again, a relation $\succeq$ on the set of all nonempty and finite subsets $\mathcal{X}$ of X is to be established but, rather than thinking of the elements of $\mathcal{X}$ as sets of uncertain outcomes, we interpret these sets as opportunity sets or menus from which an agent can make a choice. Because of this difference in interpretation, some of the axioms that can be considered reasonable in this framework differ substantially from some of those introduced in Section 3.

The first conceptual issue we want to address in this section is why one might want to deviate from the indirect-utility criterion at all in establishing a ranking of opportunity sets. The indirect-utility ranking $\succeq_U$ of opportunity sets is defined by letting, for all $A, B \in \mathcal{X}$,

$$A \succeq_U B \Leftrightarrow \max(A) R \max(B).$$

That is, only the best elements according to R in the sets to be compared matter in establishing an ordering on $\mathcal{X}$. Kreps [89] provides a characterization of the indirect-utility criterion in a model where R is not fixed. The axiom used in this characterization is the following extension-robustness condition. It requires that adding a set B that is at most as good as a given set A to A leads to a set that is indifferent to A itself.

Extension Robustness: For all $A, B \in \mathcal{X}$,

$$A \succeq B \Rightarrow A \sim A \cup B.$$

An ordering $\succeq$ on $\mathcal{X}$ satisfies extension robustness if and only if there exists an ordering R on X such that $\succeq$ is the indirect-utility ranking for the ordering R; see Kreps [89]. Clearly, this result is not a characterization of $\succeq$ for a fixed ordering R on X—note that the extension-robustness axiom does not make any reference to an underlying ordering.

The indirect-utility criterion is based on the position that the quality of the final choice of the agent is all that matters, and the only reason other characteristics of an opportunity set might be of interest is that they may have instrumental value in achieving as high a level of well-being as possible. Although this welfarist view has strong intuitive appeal, there may be situations where one might want to go beyond this criterion when assessing opportunity sets.

For instance, the way alternatives are formulated in economic models is often very restrictive, and the 'consumption' of these alternative may not capture everything of value to an agent. In this case, utility is not an indicator of overall well-being but, rather, a measure of one aspect of well-being. See, for example, Griffin [69] for a comprehensive account of well-being and its determinants. Considering criteria other than indirect utility when ranking opportunity sets can then be interpreted as an attempt to gain more information about the actual preferences, taking into account aspects in addition to the utility derived from experiencing the ultimately chosen alternative. This position is still compatible with a welfarist approach; see, for example, Bossert [26] for a discussion of this interpretation.

Another view is to assign intrinsic value to aspects other than indirect utility, in which case a departure from the welfarist viewpoint is entailed. A more detailed discussion of this position can be found, for instance, in Sen [136, 138] and in Sugden [143].

We do not want to come down firmly in favour of one of those possible justifications for examining criteria other than indirect utility for ranking opportunity sets but, rather, note that different arguments can be appealed to when motivating the approaches surveyed in this section.

Once we deviate from the position that indirect utility is all that matters, a variety of other criteria emerge as possible candidates to influence the structure of a ranking $\succeq$ on $\mathcal{X}$, and we present several approaches associated with some specific interpretations of this ranking in the remainder of this section. For example, we might want to rank opportunity sets in terms of the freedom of choice they offer (Subsection 4.2), in terms of their overall contribution to individual well-being (Subsection 4.3), or in terms of the flexibility regarding possible future preferences or possible consequences associated with the choice of specific menu items (Subsection 4.4). See also Foster [58] for a survey of the literature on the assessment of opportunities and their relationship to freedom and well-being.

4.2 Freedom of Choice

The ranking of opportunity sets in terms of the degrees of freedom that they offer to an agent has been the subject of considerable axiomatic analysis in recent years. The earliest axiomatic contribution that we know of in this area is due to Jones and Sugden [77]. They were the originators of many ideas explored, nearly a decade later, by several writers such as Pattanaik and Xu [114] who were unaware of this fundamental earlier contribution. Before surveying specific technical contributions in this literature, we first discuss briefly some of the conceptual issues involved.

Much of the literature with which we are concerned conceives an agent's opportunity set as the set of all feasible (mutually exclusive) options, from which the agent can have any option by simply choosing to have it. The literature also typically assumes that the agent's opportunity set is given. The focus is on the issue of ranking different conceivable opportunity sets in terms of the different degrees of freedom that they offer to the agent, and the analysis does not normally investigate the determinants of the opportunity set or the substantive content of the notion of alternatives belonging to an opportunity set. While the abstraction involved in this mode of approaching the problem is understandable, it is important to note that it removes from the scope of the analysis certain important considerations. For example, suppose the opportunity set is the set of all possible careers that the agent can possibly follow, and suppose a given opportunity set A for the agent does not include a career as a football player. Then, in assessing the freedom associated with the opportunity set A, it may be relevant to ask why this career is not open to the agent. It is possible to argue that, if the career as a football player is ruled out because the agent happens to be black and the state has decreed that a black person cannot play in a football team, then the loss of freedom for the agent, involved in the exclusion of this specific option, is greater as compared to a situation where the career as a football player is ruled out for the agent because of some physical handicap. Such considerations cannot be taken into account when the opportunity set is assumed to be given as the consequence of some unspecified set of factors. There are also other conceptual problems involved in identifying the opportunity set of an individual in a multi-person society, where the action of each individual affects what the other individuals can or cannot achieve. For example, in the classical economic model of a two-person, two-commodity, pure exchange economy with private ownership, it is difficult to specify, in a useful fashion, an individual's opportunity set as a set of consumption bundles any one of which can be chosen by the individual at its own discretion (in this framework, the only commodity bundles that an individual can consume entirely at its own discretion, that is, independently of the other individual's action, are its initial endowment bundle and any other commodity bundle that does not contain more of any commodity than the initial endowment bundle). We will come back to this problem later in this subsection. It is not our intention to suggest that one must settle all these conceptual issues before undertaking any analysis of the problem of ranking opportunity sets in terms of freedom. However, in assessing the literature, it is useful to keep in mind that the practice of taking as one's starting point opportunity sets, which are assumed to be given somehow, does ignore some fairly difficult and important issues.

Now, confining oneself to the problem of ranking opportunity sets which are assumed to be given, one may like to distinguish between two interpretations of such a ranking. Given two opportunity sets $A, B \in \mathcal{X}$, one may ask whether, under some particular conception of freedom, A offers more freedom than B. Alternatively, given that A gives more freedom to the agent than B under some particular conception of freedom, one can ask whether this extra freedom that A offers has any value when the value of freedom is conceived in some specific fashion. This is the distinction that van Hees [145] has drawn between the "value we attach to (different conceptions of) freedom and the contents of (different conceptions of) freedom" (see also Carter [33] and Oppenheim [107]). Thus, without using our ordinary language in too fanciful a fashion, one may say that, under some particular conception of freedom, the opportunity set A offers more freedom to the agent than the opportunity set B, but the extra freedom offered by A does not have any value for the agent (under some particular conception of the value of freedom). Some writers (see, for example, Foster [57]) have sought to capture the distinction by using the term effective freedom as distinct from the term freedom. Of course, the notion of effective freedom has to be based on some conception of what constitutes the value of freedom, so that effectiveness of freedom can have some substantive content.

Suppose one wants one's ranking of opportunity sets to reflect effective freedom, that is, freedom which has some value under some conception of the value of freedom. What are the different important conceptions of the value of freedom that one may like to consider in this context? Here a multitude of alternative routes are available. We give below a list of five different positions regarding the value of freedom, but the list is by no means exhaustive.

Position 1 The value of freedom lies in the utility that it enables the agent to realize, the utility being judged in terms of the present preferences of the agent. This is, of course, the indirect-utility approach discussed in the previous subsection. Under this conception of the value of freedom, an opportunity set A is ranked at least as high as another opportunity set B if and only if, in terms of the present preferences of the agent, the agent's most preferred alternative in A is at least as good as the agent's most preferred alternative in B.

Position 2 The value of an opportunity set lies in the flexibility that it provides to the agent. The reason flexibility is valued is that the agent is uncertain about its own preference at the time it would have to make its choice. Therefore, the greater the flexibility embodied by the opportunity set, the better it is for the agent. While this flexibility-based conception of the value of freedom is different from the conception based on indirect utility, intuitively it still centres around the agent's preferences: the agent's preferences which are relevant here are the uncertain future preferences. The contributions of both Kreps [89] and Arrow [6] are based on this conception of the value of freedom; see Subsection 4.4.

Position 3 A liberal tradition going back to Mill [96] regards the diversity of the society as a desirable feature in itself, and considers the individuals' freedom to choose to be an effective means for ensuring such diversity in the long run.

Position 4 We also have the argument, again traceable to Mill [96], that the act of choosing from among alternative options contributes to the development of the human faculties of an individual. For this argument to be intuitively plausible, the choice must be significant (see Sugden [143]): the choice between a good apple and a rotten apple is not significant because the choice is obvious (in a sense to be explored later) and one does not have to exercise much of one's faculty to choose between the two.

Position 5 Yet another important strand of libertarian thought respects the ability of people to choose for themselves, since it is only by making choices regarding the various aspects of its own life that an individual can live a meaningful life. Thus, as Nozick [104] writes, "A person's shaping his life in accordance with some overall plan is his way of giving meaning to his life; only a being with the capacity so to shape his life can have or strive for meaningful life."

Positions 1 to 4 in this list are consequentialist in nature, insofar as they consider freedom as an instrument for achieving something (utility, social diversity, the development of human faculties, etc.), while position 5 is concerned with the intrinsic value of freedom. For a detailed discussion of positions 3, 4, and 5, the reader may refer to Jones and Sugden [77] and also Sugden [143].

The conceptual basis of many of the formal models on the ranking of opportunity sets in terms of freedom is to be found in the belief that freedom of choice has a value that is independent of the amount of utility that may be generated by such freedom. Suppose we feel that freedom of choice has a value for some of the three non-welfarist reasons given above. Then one can think of two distinct aspects of an opportunity set, which may be relevant. The first aspect relates to the volume of options figuring in the opportunity set. The second aspect relates to the significance of these options in the opportunity set, where significance is to be judged in terms of the reasons that we have enumerated above.

Let us first consider the volume of options. Steiner [142], Pattanaik and Xu [114, 117], and van Hees [145] are some of the contributions which seek to capture the volume-of-options aspect of opportunity sets. When the number of options is finite, the simplest way of assessing the volume or quantity of options available to the agent is to count how many options there are. This is indeed the notion of volume of options that underlies both Steiner [142] and Pattanaik and Xu [114], though there are differences between the two contributions. Steiner's starts with a given (finite) list of actions that the agent may or may not be free to take. Given such a list, Steiner would count the number n of actions in the list that the agent is free to take, and the number of actions $\bar{n}$ of actions in the list that the agent is not free to take. He would then use the ratio $n/(n + \bar{n})$ as the index of the agent's freedom. Steiner seeks to justify this measure by countering various possible objections to it.

Pattanaik and Xu [114] follow an axiomatic approach to the problem, and use three axioms to characterize a rule for ranking finite opportunity sets on the basis of their cardinalities. This cardinality-based ordering $\succeq_C$ on $\mathcal{X}$ is defined by letting, for all $A, B \in \mathcal{X}$,

$$A \succeq_C B \Leftrightarrow |A| \geq |B|.$$

Pattanaik and Xu [114] characterize $\succeq_C$ in terms of the following axioms. First, they require that, if neither of two sets offer any freedom of choice at all, in the sense that each of them contains exactly one alternative, then the two sets should be indifferent in terms of freedom. If the interpretation of the ranking to be established is in terms of freedom of choice, as is the case in this subsection, this requirement has intuitive appeal. Clearly, if X contains at least two elements, this axiom is incompatible with the extension-rule requirement introduced in the previous section. Jones and Sugden [77] were the first to introduce this axiom, which they called the principle of no choice. Pattanaik and Xu [114], unaware of this earlier contribution, introduced the axiom much later.

Indifference Between No-Choice Situations: For all $x, y \in X$,

$$\{x\} \sim \{y\}.$$

The next condition is a monotonicity requirement. If a new alternative is added to a singleton, the resulting two-element set is considered better than the singleton because the former offers some freedom of choice whereas the latter does not.

Simple Expansion Monotonicity: For all distinct $x, y \in X$,

$$\{x, y\} \succ \{x\}.$$

The final axiom is an independence condition which strengthens the strict independence axiom introduced in Section 3.

Strong Independence: For all $A, B \in \mathcal{X}$, for all $x \in X \setminus (A \cup B)$,

$$A \succeq B \Leftrightarrow A \cup \{x\} \succeq B \cup \{x\}.$$

The intuitive motivation of this axiom is based on a separability argument and is, in spite of the different interpretation considered in this section, analogous to the one that can be given in support of strict independence.

We now obtain the following characterization of $\succeq_C$, due to Pattanaik and Xu [114]. Note that it is sufficient to assume that $\succeq$ is transitive—reflexivity and completeness follow as consequences of this assumption and the axioms.

Theorem 14 *Suppose $\succeq$ is a transitive relation on $\mathcal{X}$. $\succeq$ satisfies indifference between no-choice situations, simple expansion monotonicity, and strong independence if and only if $\succeq = \succeq_C$.*

Proof. It is straightforward to verify that $\succeq_C$ satisfies the required axioms. Now suppose $\succeq$ is a transitive relation on $\mathcal{X}$ satisfying indifference between no-choice situations, simple expansion monotonicity, and strong independence. Because $\succeq_C$ is complete, it is sufficient to prove that, for all $A, B \in \mathcal{X}$,

$$|A| = |B| \Rightarrow A \sim B \tag{17.27}$$

and

$$|A| > |B| \Rightarrow A \succ B. \tag{17.28}$$

We prove (17.27) by induction over the number of elements in A and B. If $|A| = |B| = 1$, the claim follows immediately from indifference between no-choice situations. Now suppose (17.27) is true for all $A, B \in \mathcal{X}_n$ with $1 \leq n < |X|$. Consider $A, B \in \mathcal{X}_{n+1}$. Let $C \subset A$ be such that $|C| = n$. It follows that $A \setminus C = \{x\}$ for some $x \in A$. We can distinguish two cases.

(i) $x \in B$;

(ii) $x \notin B$.

In case (i), let $D = B \setminus \{x\}$. Because $|C| = |D| = n$, the induction hypothesis implies that $C \sim D$, and strong independence implies $A = C \cup \{x\} \sim D \cup \{x\} = B$.

Now consider case (ii). Because $x \notin B$ and $|A| = |B|$, it follows that $B \setminus A \neq \emptyset$. Let $E = B \setminus \{y\}$ for some $y \in B \setminus A$. Because $|C| = |E| = n$, the induction hypothesis implies that $C \sim E$, and strong independence implies

$$A = C \cup \{x\} \sim E \cup \{x\}. \tag{17.29}$$

Let $z \in E$. Because $|(E \cup \{x\}) \setminus \{z\}| = |B \setminus \{z\}| = n$, the induction hypothesis implies $(E \cup \{x\}) \setminus \{z\} \sim B \setminus \{z\}$, and strong independence implies $E \cup \{x\} \sim B$. By (17.29) and transitivity, it follows that $A \sim B$.

To prove (17.28), suppose $A, B \in \mathcal{X}$ are such that $|A| > |B|$. Let $F \subset A$ be such that $|F| = |B|$. By (17.27),

$$F \sim B. \tag{17.30}$$

Let $G = A \setminus F$, and let $f \in F$ and $g \in G$. By simple expansion monotonicity, $\{f, g\} \succ \{f\}$. Using strong independence and transitivity repeatedly, we obtain $A = F \cup G \succ F$ and, together with (17.30) and transitivity, it follows that $A \succ B$. ■

The idea of ranking opprtunity sets simply on the basis of their cardinalities has been found too naive by many people including Pattanaik and Xu [114] themselves. It is, therefore, natural that the axioms underlying the result of Pattanaik and Xu have been the subject of searching criticism. We consider some of these criticisms below. However, before we take up these issues, it may be appropriate to note a somewhat different problem with the cardinality-based approach.

In the case of finite sets, one can use the cardinality of the set as a natural index of the quantity of options in the set. However, in the case of infinite sets, no such simple index of the quantity of options is available. Yet, in many economic contexts, the agent's opportunity set is typically an infinite set—consider, for example, a consumer's budget set in standard neoclassical demand theory. The quantity-based approach to the issue of ranking infinite opportunity sets has been explored by Pattanaik and Xu [117]; Laslier [91] also discusses infinite opportunity sets. Pattanaik and Xu [117] consider the problem of ranking all compact subsets of a given real space. They introduce four properties of a freedom ranking (one of these properties implies indifference between no-choice situations) and show that these four properties together are sufficient to ensure that the freedom ordering on the class of all compact subsets of the m-dimensional real space has a countably additive real-valued representation with some appealing properties. The ranking of the different compact subsets of $\mathbb{R}^m$ based on their volumes is an example of a ranking which satisfies all the four properties of Pattanaik and Xu [117]; the volume clearly provides a countably additive real-valued representation of the type mentioned above.

One basic criticism of Pattanaik and Xu's [114] axioms came from Sen [137, 138] who argued that indifference between no-choice situations was not an acceptable axiom because it ignored the vital role of preferences in judgements regarding freedom or opportunity. Sen [137] wrote:

> "Suppose I wish to go home from the office by taking a short walk. Consider now two alternatives: (1) I can hop on one leg to home, but I am not permitted to walk, and (2) I can walk normally to home, but I am not permitted to hop on one leg. Given my preferences (in the sense of what I would choose given the choice) it would be absurd to say that I have exactly as much effective freedom in the first case (i.e., hop, not walk) as in the second (i.e., walk, not hop)."

Thus, the argument is that, since the agent would prefer to walk normally rather than to hop on one leg, the opportunity set which contains only the option of walking normally is better than the opportunity set containing only the option of hopping on one leg. A similar, preference-based argument can be advanced against simple expansion monotonicity too (see Sen [138]). For instance, conforming to the spirit of Sen's example mentioned above, one can argue that if, to the opportunity set that contains the sole option of walking home normally, one adds the option of hopping home on one leg, one does not really increase the freedom of the agent. Sen's [138] basic point is that the "idea of effective freedom cannot be disassociated from our preferences." A related issue is that having a number of similar alternatives available may not provide the same degree of freedom as having the same number of distinct options. In order to formalize this notion in more precise terms, a definition of dissimilarity or diversity is required. See, for example, Klemisch-Ahlert [81] and Pattanaik and Xu [116] for discussions.

The notion of effective freedom leads to the question, effective for what?, and thus brings us back to the issue of what constitutes the value of freedom and to what extent preferences are relevant in assessing this value. The issue has been incisively discussed by Jones and Sugden [77] (some of the ideas of Jones and Sugden [77] were to be refined later by Sugden [143]). One of the most interesting aspects of the contribution of Jones and Sugden [77] is that they provide a persuasive account of the role of preferences in the assessment of freedom without making the value of freedom directly dependent on the preference of the agent under consideration. This is in contrast to Sen [137, 138] who seems to emphasize that the preferences relevant in assessing an agent's freedom are the preferences of the agent itself.

For Jones and Sugden [77] and Sugden [143], the preferences to be taken into account in assessing opportunity sets are the preferences that reasonable people would have in the position of the agent whose freedom we are assessing. Thus, Jones and Sugden [77] are concerned with a set $\mathcal{R}$ of preference orderings, the interpretation of $\mathcal{R}$ being that it is the set of all possible preference orderings on X that a reasonable person may have if placed in the agent's objective position. In principle, $\mathcal{R}$ may contain only one ordering (if the situation is such that, when placed in the agent's objective position, a reasonable person can have only one ordering) but, in general, this need not be the case. Note that the agent's preference ordering may not belong to $\mathcal{R}$. This, of course, begs the question about who can be considered to be a reasonable person, that is, how one can identify $\mathcal{R}$. However, Jones and Sugden [77] and Sugden [143] take $\mathcal{R}$ to be given and focus on the analytical use of the notion of an ordering that a reasonable person, in the agent's position, may have.

Jones and Sugden [77] introduce three axioms. One of them is indifference between no-choice situations. The other two axioms use the notion of an option which is significant in relation to a given opportunity set containing the option. Let $A \in \mathcal{X}$ and $x \in A$. x is significant in relation to A if and only if there exists an ordering $R \in \mathcal{R}$ such that x is the unique best element in A according to R. Thus, an option $x \in A$ is significant in relation to A if and only if some reasonable person, in the agent's position, would consider x to be strictly better than every other alternative in A.

Principle of Addition of Significant Options: For all $A \in \mathcal{X}$, for all $x \in X \setminus A$,

$$x \text{ is significant in relation to } A \cup \{x\} \Rightarrow A \cup \{x\} \succ A.$$

Principle of Addition of Insignificant Options: For all $A \in \mathcal{X}$, for all $x \in X \setminus A$,

$$x \text{ is not significant in relation to } A \cup \{x\} \Rightarrow A \cup \{x\} \sim A.$$

The principle of addition of significant options tells us that, if we add to a set an alternative which some reasonable person would strictly prefer to every alternative in the set, then the expanded set offers more freedom than the original set. The principle of addition of insignificant options tells us that, if we add to a given set an alternative such that no reasonable person would strictly prefer the alternative to every alternative in the set, then the expanded set offers the same amount of freedom as the original set. An elegant result due to Jones and Sugden [77] shows that, if there exist two alternatives x and y in X such that every reasonable person would strictly prefer x to y, then no ranking rule can simultaneously satisfy indifference between no-choice situations, the principle of addition of significant options, and the principle of addition of insignificant options.

Theorem 15 *Suppose there exist $x, y \in X$ such that xPy for all $R \in \mathcal{R}$. There exists no transitive relation $\succeq$ on $\mathcal{X}$ satisfying indifference between no-choice situations, the principle of addition of significant options, and the principle of addition of insignificant options.*

Proof. Let $x, y \in X$ be such that xPy for all $R \in \mathcal{R}$. Then, by the priciple of addition of significant options, $\{x, y\} \succ \{y\}$; by indifference between no-choice situations, $\{y\} \sim \{x\}$; and, finally, by the principle of addition of insignificant options, $\{x\} \sim \{x, y\}$. But this violates the transitivity of $\succeq$. ■

A possible interpretation of Theorem 15 is as follows. Indifference between no-choice situations and the principle of addition of insignificant options seem to have their intuitive basis in Mill's non-utilitarian, but consequentialist, argument that significant choice develops human character, or in Nozick's argument that a person can live a meaningful life only by making significant choices with respect to various aspects of its life. On the other hand, as Sugden [143] argues persuasively, the principle of addition of significant options is not compatible with the intuition underlying these approaches to the value of freedom. An individual may have to do a lot of thinking to choose from a given set of three options, since none of them may be obviously better than the rest. However, if one adds to the set another option which is obviously better than each of the three original options, then there may not be any more need for the exercise of human faculties to make a choice from the set of four options. Therefore, the expanded set may offer less scope for significant choosing than the original set.

Though Jones and Sugden [77] interpret $\mathcal{R}$, the reference set of preference orderings, as the set of all possible preference orderings that a reasonable person may have in the agent's objective situation, one can think of other plausible interpretations of this set. For example, one can interpret $\mathcal{R}$ as the set of all preference orderings that the individuals in the society actually have. Without committing himself to any specific interpretation of $\mathcal{R}$, Foster [57] investigates the properties of the unanimity relation $\succeq_N$ on $\mathcal{X}$, defined as follows. For all $A, B \in \mathcal{X}$,

$$A \succeq_N B \Leftrightarrow \text{ for all } R \in \mathcal{R}, \exists x \in \bar{\mathcal{B}}(A, R), y \in \bar{\mathcal{B}}(B, R) \text{ such that } xRy.$$

Thus, A offers at least as much freedom as B if and only if, in terms of every preference figuring in the reference set $\mathcal{R}$, A offers at least as much indirect utility as B. The statement $A \succeq_N B$ can be interpreted as a sufficient condition for saying that A offers at least as much freedom as B. Alternatively, one could interpret $A \succeq_N B$ as a necessary and sufficient condition for saying that A offers at least as much freedom as B. The latter interpretation would seem rather strong: it is not difficult to think of situations where we have neither $A \succeq_N B$ nor $B \succeq_N A$, but one would like to say that A offers at least as much freedom as B. The former interpretation, which is more plausible, is inconsistent with indifference between no-choice situations if there are at least two options $x, y \in X$ such that xPy for every $R \in \mathcal{R}$.

$\succeq_N$ is a quasi-ordering but not necessarily an ordering. Foster [57] defines an ordering on X which constitutes an interesting extension of $\succeq_N$. For all $A \in \mathcal{X}$, let $\hat{\mathcal{X}}(A) = \{B \in \mathcal{X} \mid A \succeq_N B\}$ and $\mathcal{D}(A) = \cup_{B \in \hat{\mathcal{X}}(A)} B$. Now define the relation $\succeq_F$ on $\mathcal{X}$ as follows. For all $A, B \in \mathcal{X}$,

$$A \succeq_F B \Leftrightarrow |\mathcal{D}(A)| \geq |\mathcal{D}(B)|.$$

It can be easily checked that $\succeq_F$ is an ordering on $\mathcal{X}$ and that, for all $A, B \in \mathcal{X}$,

$$[A \succeq_N B \Rightarrow A \succeq_F B] \text{ and } [A \succ_N B \Rightarrow A \succ_F B].$$

While $\succeq_F$ provides an interesting rule for ranking opportunity sets, at this stage we do not have an axiomatic characterization of this ranking rule.

The multi-preference approach of Jones and Sugden [77] and Foster [57] has been followed by other writers. Pattanaik and Xu [115] take up the problem of ranking finite opportunity sets taking into account a reference set $\mathcal{R}$ of preference orderings, where, as in Jones and Sugden [77], $\mathcal{R}$ is interpreted as the set of all preference orderings that a reasonable person may have in the objective situation of the agent whose freedom is under consideration. For $A \in \mathcal{X}$, let $\mathcal{M}(A)$ denote the set of all alternatives $x \in A$ such that, for some $R \in \mathcal{R}$, x is a best element in A according to R. Furthermore, for all $A, B \in \mathcal{X}$, let $\bar{\mathcal{M}}(A, B)$ denote the set of all $x \in A$ such that, for all $R \in \mathcal{R}$, there exists $y \in B$ such that yPx. Pattanaik and Xu [115] provide characterizations of the following two orderings $\succeq_M^1$ and $\succeq_M^2$ on $\mathcal{X}$. For all $A, B \in \mathcal{X}$,

$$A \succeq_M^1 B \Leftrightarrow |\mathcal{M}(A)| \geq |\mathcal{M}(B)|$$

and

$$A \succeq_M^2 B \Leftrightarrow |\mathcal{M}(A) \setminus \bar{\mathcal{M}}(A, B)| \geq |\mathcal{M}(B) \setminus \bar{\mathcal{M}}(B, A)|.$$

Another example of a multi-preference approach to the problem of ranking opportunity sets is presented in Pattanaik and Xu [117]. Pattanaik and Xu [117] consider the problem of ranking all compact subsets of the m-dimensional real space and, in many ways, the formal strcture of their multi-preference approach to this problem parallels that of their quantity-based approach to the same problem, which we have mentioned earlier.

Puppe's [120] paper is an important contribution that takes into account the quality of options, as distinct from the volume of options, in ranking opportunity sets, but does so in a fashion very different from the multi-preference approach followed by Jones and Sugden [77], Foster [57], and Pattanaik and Xu [115, 117]. The central axiom in Puppe's [120] analysis is the axiom of preference for freedom of choice. This approach involves including the empty set as one of the sets to be ranked. We define $\mathcal{X}_0 = \mathcal{X} \cup \{\emptyset\}$ and use the convention that $A \succ \emptyset$ for all $A \in \mathcal{X}$. Puppe's [120] axiom is defined as follows.

Preference for Freedom of Choice: For all $A \in \mathcal{X}$, there exists $x \in A$ such that

$$A \succ A \setminus \{x\}. \tag{17.31}$$

An alternative $x \in A$ such that (17.31) is satisfied is called essential in A. An essential element in an opportunity set may be interpreted in many ways. First, assuming that the agent has a preference ordering R on X, one can interpret an essential element in $A \in \mathcal{X}$ as a best element in A according to R. Second, using the multi-preference approach of Jones and Sugden [77], Foster [57], and Sugden [143], one can interpret an essential element in A as a best element in A according to some $R \in \mathcal{R}$.

For all $A \in \mathcal{X}$, let $\mathcal{E}(A)$ denote the set of all $x \in A$ such that x is essential in A. If $\succeq = \succeq_C$, then, for all $A \in \mathcal{X}$, we have $A = \mathcal{E}(A)$. Puppe [120] introduces the following two plausible axioms. The first of these requires every opportunity set $A \in \mathcal{X}$ to be indifferent to the set of all essential alternatives in A, and the second requires every opportunity set to be at least as good as every subset of itself.

Independence of Non-Essential Alternatives: For all $A \in \mathcal{X}$,

$$A \sim \mathcal{E}(A).$$

Monotonicity with Respect to Set Inclusion: For all $A, B \in \mathcal{X}$,

$$B \subseteq A \Rightarrow A \succeq B.$$

These three axioms are used by Puppe [120] to clarify the structure of a domination relation induced by $\succeq$, which says that an opportunity set $A \in \mathcal{X}$ dominates a set $B \in \mathcal{X}$ in that sense if and only if $A \succeq A \cup B$—that is, adding B to A does not lead to a better set acocrding to $\succeq$. The following result is due to Puppe [120] who states it in a slightly different but equivalent form.

Theorem 16 *Suppose $\succeq$ is an ordering on $\mathcal{X}$ satisfying preference for freedom of choice and monotonicity with respect to set inclusion. $\succeq$ satisfies independence of non-essential alternatives if and only if, for all $A, B \in \mathcal{X}$,*

$$A \succeq A \cup B \Leftrightarrow \mathcal{E}(A \cup B) \subseteq B.$$

Proof. See Puppe [120]. ■

Theorem 16 implies that, given the three axioms in the statement, the domination relation induced by $\succeq$ is completely determined by the correspondence $\mathcal{E}$. The class of orderings satisfying preference for freedom of choice, monotonicity with respect to set inclusion, and independence of non-essential alternatives is very large. This is the case because, first, no restriction other than $\mathcal{E}(A) \subseteq A$ for all $A \in \mathcal{X}$ is stipulated for the correspondence $\mathcal{E}$ and, second, Theorem 16 does not impose any restriction on the ranking of a pair of opportunity sets when the dominance relation does not hold in either direction between two opportunity sets.

An interesting feature of Puppe's [120] treatment of the quality of options is that, intuitively, the quality of an option is 'inferred' from the ranking of the opportunity sets itself (see the above definition of an essential element in an opportunity set). In that sense, Puppe [120] treats the quality of an option as being 'revealed' by the ranking of opportunity sets, rather than conceptualizing it in terms of a primitive set of preference orderings on X, as is done in most contributions in the multi-preference framework. This naturally raises the question whether, starting with the freedom ranking $\succeq$ on $\mathcal{X}$, one can, under certain circumstances, 'infer' the existence of a set $\{R_1, \ldots, R_n\}$ of preference orderings on X which are linked to $\succeq$ in a plausible fashion. Using a fundamental theorem due to Kreps [89], Nehring and Puppe [100] and Puppe [121] show that one can indeed do this if the ranking on $\mathcal{X}$ satisfies certain intuitively appealing conditions. Consider the following axiom.

Contraction Consistency: For all $A, B \in \mathcal{X}$, for all $x \in X \setminus A$,

$$[B \subseteq A \text{ and } A \cup \{x\} \succ A] \Rightarrow B \cup \{x\} \succ B.$$

Contraction consistency tells us that if adding an option x to an opportunity set A increases freedom, then so must the addition of that option to any subset of A. Using the theorem of Kreps [89], Nehring and Puppe [100] and Puppe [121] show that, if X is finite and $\succeq$ satisfies montonicity with respect to set inclusion, then $\succeq$ satisfies contraction consistency if and only if there exists a set of orderings $\{R_1, \ldots, R_n\}$ on X such that, for all $A, B \in \mathcal{X}$,

$$A \sim A \cup B \Leftrightarrow [\text{ for all } i \in \{1, \ldots, n\} \text{ and all } b \in B, \exists a \in A \text{ such that } aR_ib]. \qquad (17.32)$$

Note that the set of orderings $\{R_1, \ldots, R_n\}$ such that (17.32) is satisfied need not be unique.

The analysis of Nehring and Puppe [100] and Puppe [121] thus provides us with an 'indirect' multi-preference framework where, given that $\succeq$ satisfies monotonicity with respect to set inclusion and contraction consistency, one can infer the existence (but not the uniqueness) of a set of preference orderings on X. This is distinct from the 'direct' multi-preference approach of Jones and Sugden [77], Sugden [143], Foster [57], and Pattanaik and Xu [115, 117], where one starts with a given set of preference orderings on X.

Irrespective of whether one uses the indirect multi-preference framework and infers the existence of a set of preference orderings on X, or the direct multi-preference approach that starts with a given set of preference orderings on X, the passage from the preference orderings on X to the freedom ranking on $\mathcal{X}$ can be interpreted in terms of the standard social choice problem of deriving a social ordering from a profile of individual orderings. In the multi-preference approach to the ranking of opportunity sets, the freedom ranking is defined on $\mathcal{X}$, whereas in standard social choice theory, the social ranking to be determined by aggregating individual orderings on X is defined on the same set X. This difference can be removed by considering, for each $i \in \{1, \ldots, n\}$, the indirect utility ordering as defined in Subsection 4.1. One can then view the problem as one of aggregating the profile of indirect utility orderings on $\mathcal{X}$ so as to derive the ranking $\succeq$ on $\mathcal{X}$. In his indirect multi-preference framework, Puppe [121] uses the resulting analogy to explore the structure of several specific classes of freedom rankings on $\mathcal{X}$.

So far we have been concerned with the problem of ranking alternative opportunity sets for a given agent. However, given the scarcity of resources, if a society wants to give more opportunities or greater freedom to choose to one individual, the freedom of some other individual may have to be curtailed. Furthermore, if an individual's freedom of choice, as reflected in its opportunity set, is valued by the individual or by the society, then the distribution of such freedom of choice among the individuals will also be of ethical importance. Different issues relating to the interdependence of the opportunity sets of different individuals in a society, given the scarcity of resources, and the social concern about the distribution of opportunities are discussed in a number of contributions; see also Peragine [118] for a survey of that branch of the literature.

Gravel, Laslier, and Trannoy [68] consider the problem of ranking the possible opportunity sets of an individual in a framework where there are several individuals and where the presence of aggregate scarcity limits the total number of individuals who can simultaneously have any given option in their respective opportunity sets. Intuitively, the incorporation of the consideration of scarcity makes the opportunity sets of the individuals interdependent: given scarcity, if the opportunity sets of some individuals are specified in a certain fashion, then that specification imposes restrictions on the possible specifications of opportunity sets for the remaining individuals. In addition to assuming that the set X of all conceivable options is finite and the ranking of opportunity sets is an ordering that is identical for all individuals, Gravel, Laslier, and Trannoy [68] introduce two axioms. The first axiom extends the idea underlying simple expansion monotonicity to arbitrary sets and, thus, requires that, for all $A, B \in \mathcal{X}$, if B is a proper subset of A, then A offers more freedom than B. The second axiom seeks to capture the intuition that a simple reallocation of options among the opportunity sets of different individuals cannot increase the freedom of every individual. The main result of their paper shows that, if the freedom ordering on $\mathcal{X}$ (assumed to be identical for all individuals) satisfies the above two axioms, then the options in X can be assigned numerical weights such that, for all $A, B \in \mathcal{X}$, A offers at least as much freedom as B if and only if the sum of the weights of all the options in A is no less than the sum of the weights of all the options in B.

The central conceptual feature of Gravel, Laslier, and Trannoy [68] is the notion that the scarcity of aggregate resources in the society makes the opportunity sets of different individuals interdependent. The recognition of such interdependence is an important advancement. It may, however, be worth noting here that the interdependence analyzed by Gravel, Laslier, and Trannoy [68] is very different from another type of interdependence that we will consider later in this subsection, namely, the interdependence that exists in an interactive situation where an individual can only choose an *action* and not an outcome, the outcome for a given individual being, in general, determined by the actions chosen by *all* individuals.

Many of the issues that arise in the context of the distribution of opportunities in a society are analogous to various issues in the framework of standard welfare analysis, especially when the scarcity of aggregate resources makes the opportunity sets of the individuals interdependent. Thus, we may pose questions such as the following. How efficient is the competitive market mechanism in the allocation of opportunities? Starting with a competitive equilibrium, is it possible to expand the freedom or the opportunities of one individual without contracting the freedom of some other individual? How does one measure the extent of equality in the distribution of opportunities in a society? These questions clearly are counterparts of familiar questions in the welfaristic framework.

Consider first the link between freedom and the competitive market mechanism. The two fundamental theorems of welfare economics deal with the Pareto optimality of a competitive equilibrium allocation and the achievability of a Pareto optimal allocation through a competitive equilibrium. The requirement of Pareto optimality is, of course, an entirely utility-based restriction and has very little to do directly with the freedom of the individuals in the economy. Yet, one of the traditional arguments for the market mechanism, as distinct from command-based allocation mechanisms, has been that competitive markets promote freedom. The question, therefore, naturally arises whether results analogous to the two fundamental theorems of welfare economics can be proved when one replaces the utility-based criterion of Pareto optimality by some freedom-based criterion. In particular, is it possible to show that, given some plausible notion of freedom, if one starts with a competitive equilibrium allocation, then it is impossible to increase the freedom of one consumer without reducing the freedom of another consumer? Sen [140] addresses this important issue.

In the standard framework of general-equilibrium theory, the opportunity set of a consumer can be conceived as a nonempty subset of its consumption set. Sen [140] identifies the opportunity set of a consumer in a competitive equilibrium with its budget set in that competitive equilibrium. Furthermore, Sen [140] makes the following important assumption regarding the freedom ranking of alternative opportunity sets of a consumer. For all opportunity sets A and B, A offers the consumer at least as much freedom as B if and only if B is a subset of A. Given this assumption, Sen [140] shows that a competitive equilibrium allocation is 'freedom-optimal' in the sense that, if one starts with a competitive equilibrium allocation, then it will be impossible to increase the freedom of one individual without reducing the freedom of any other individual. This, of course, is the freedom-based counterpart of the first theorem of welfare economics. Xu [148] continues with the Sen assumption regarding the freedom ranking of a consumer's opportunity sets, and extends Sen's [140] result by proving a freedom-based counterpart of the second theorem of welfare economics. While the if part of the assumption underlying the results of Sen [140] and Xu [148] is unexceptionable, the only-if part is extremely strong and, as Xu [148] shows with an example, it may not be possible to prove those results if it is relaxed.

See also Arlegi and Nieto [2], Bossert, Fleurbaey, and Van de gaer [30], Herrero [72], Herrero, Iturbe-Ormaetxe, and Nieto [73], Kranich [86, 87], Ok [105], Ok and Kranich [106] who, among others, discuss various issues relating to equality in the distribution of opportunities.

Kranich [86] is concerned with the problem of establishing an ordering in terms of equality on the set of possible distributions of finite opportunity sets among the individuals in a given society. He starts with a two-individal society and seeks to characterize a specific equality ordering on $\mathcal{X}^2$ (each element (A, B) of $\mathcal{X}^2$ represents a distribution of opportunity sets, where individual 1 has the opportunity set A and individual 2 has the opportunity set B). For a two-individual society, Kranich [86] imposes the following axioms on an equality ordering $\succeq^e$ on $\mathcal{X}^2$.

Anonymity: For all $(A, B) \in \mathcal{X}^2$,
$$(A, B) \sim^e (B, A).$$

Montonicity of Equality: For all $A, B, C \in \mathcal{X}$ such that $A \subseteq B \subset C$,
$$(A, B) \succ^e (A, C).$$

Independence of Common Expansions: For all $A, B, C \in \mathcal{X}$ such that $C \cap (A \cup B) = \emptyset$,
$$(A \cup C, B \cup C) \sim^e (A, B).$$

Assimilation: For all $(A, B) \in \mathcal{X}^2$, for all $a \in A$, $b \in B$, $c \in (X \setminus [(A \setminus \{a\}) \cup (B \setminus \{b\})])$,
$$((A \setminus \{a\}) \cup \{c\}, (B \setminus \{b\}) \cup \{c\}) \succeq^e (A, B).$$

Anonymity requires that if the two individuals exchange their opportunity sets, then the equality of distribution is not affected. Monotonicity of equality requires that if the opportunity set of individual 1 is a subset of that of individual 2, then an expansion of the opportunity set of 2, without any change in the opportunity set of 1, reduces the degree of equality. Anonymity and monotonicity of equality are highly plausible axioms. However, the axiom of independence of common expansions is less plausible. It requires that if we discard some options common to the opportunity sets of both individuals, then the degree of equality remains unchanged. Similarly, assimilation can be challenged as well. This axiom requires that if we replace an arbitrary element $a \in A$ and an arbitrary element $b \in B$ with an option c that is neither in $A \setminus \{a\}$ nor in $B \setminus \{b\}$, then the degree of equality cannot decrease. Kranich [86] shows that the only equality ordering $\succeq^e$ on $\mathcal{X}^2$ satisfying the above four axioms is the cardinality-difference ordering $\succeq^e_{CD}$ defined as follows. For all $(A, B), (A', B') \in \mathcal{X}^2$,
$$(A, B) \succeq^e_{CD} (A', B') \Leftrightarrow |\,|A| - |B|\,| \le |\,|A'| - |B'|\,|.$$
Kranich [86] also considers societies with an arbitrary number of individuals and, in this general setting, proves a theorem regarding a real-valued representation of $\succeq^e$.

In Kranich [86], the extent of opportunities available to an individual is measured by the cardinality of its opportunity set and, in a two-individual society, the difference between the cardinalities of the two opportunity sets constitutes an index of inequality. In contrast to Kranich [86], Herrero, Iturbe-Ormaetxe, and Nieto [73] focus on another aspect of one's intuition about the equality of opportunities (see also Bossert, Fleurbaey, and Van de gaer [30] and Kranich [86]). For a two-person society, one can plausibly argue that the larger the number of options that are common between the opportunity sets of two individuals, the higher the degree of equality. More generally, for a society with any number of individuals, this argument extends to the view that the larger the number of options that are common to the opportunity sets of all individuals, the higher the degree of equality. It is this intuition that is explored in Herrero, Iturbe-Ormaetxe, and Nieto [73]. Arlegi and Nieto [2] axiomatically investigate lexicographic combinations of two principles in the two-person case. The two criteria to be combined are the cardinality of the difference of the two opportunity sets and the cardinality of their intersection. They also consider various n-person extensions of these principles.

Ok and Kranich [106] also consider the issue of the equality of a distribution of opportunity sets. They focus on the case of a two-individual society where, for each individual, the alternative opportunity sets are ranked on the basis of their cardinalities. In this framework, they prove an analogue of a basic theorem in the literature on the measurement of income inequality. They first introduce the notion of an equalizing transformation of a given pair of opportunity sets in their two-person society and also the notion of a Lorenz quasi-ordering on the set of pairs of opportunity sets. The main result of Ok and Kranich [106] shows that, in their assumed framework, one distribution of opportunity sets Lorez dominates another distribution if and only if the first distribution can be reached from the second by a finite sequence of equalizing transformations and if and only if every inequality-averse social welfare functional ranks the first distribution higher than the second. The Lorenz quasi-ordering that Ok and Kranich [106] generate in a two-individual society does not satisfy Kranich's [86] highly plausible axiom, monotonicity of equality. This, of course, is a problem similar to one that arises in the context of income distribution. Given this problem, Ok and Kranich [106] explore various extensions, satisfying monotonicity of equality, of their Lorenz quasi-ordering.

A general result due to Ok [105] has a pessimistic message regarding the possibility of measuring inequality in the distribution of opportunities. Ok [105] formulates the counterpart of the fundmental concept of an equalizing transfer familiar in the literature on income distribution (see Dalton [38]), and he does this in a way that is more general than the corresponding formulation in Ok and Kranich [106]. The formulation of the notion of an equalizing transfer here has to capture the intuition that a 'transfer' of opportunities from a person with 'more' opportunities to a person with 'fewer' opportunities increases equality, provided that such a transfer does not reverse the ranking of the two individuals in terms of the opportunities available to them. Therefore, any such formulation has to be based on some ranking of the opportunity sets that gives us an ordinal measure of the 'amount' of opportunities refleced in an opportunity set. Ok [105] introduces certain plausible forulations of an equalizing transfer with respect to opportunities, and shows that the only ranking of opportunity sets that can serve as a basis of the notion of an equalizing transfer, as formulated by him, must be the cardinality-based ranking. Since the requirements that Ok [105] imposes on the concept of an equalizing transfer of opportunities are intuitively very appealing, his central results have a strong negative flavour, given the restrictive nature of the cardinality-based ranking of opportunity sets.

We conclude this subsection by discussing the notion of an opportunity set in some detail. As mentioned earlier, the opportunity set of an agent is usually interpreted as the set of all options which are available to the agent, and any element of this set can be chosen by the agent should it wish to do so. While this may be a useful starting point, this conception, by itself, is not sufficiently rich to be able to serve as a vehicle for certain types of intuition relating to freedom, such as the intuition underlying the notion of negative freedom. The conception also seems to be limited in another way. In many situations involving interaction of individuals, it is not always clear how to specify the opportunity set as the set of all outcomes which are actually available to the individual for choice. Both these limitations have been noted in the literature, though most contributions use the notion of an opportunity set as described above.

The first limitation arises from the fact that the widely used conception of an opportunity set does not distinguish between the different types of reasons because of which an option may not be feasible for the agent. This limitation is reflected in the example that we have already given earlier. If the set of careers available to an individual does not include a career as a football player, then, for some conceptions of freedom, the assessment of freedom may depend on whether the career is ruled out by an external constraint imposed by some other agent, such as the state, or whether it is ruled out because of natural factors. While one may agree with MacCallum [93] that freedom always refers to freedom from some factor or circumstance to do or not to do or to become or not to become something, one can still subscribe to the view that whether the factor or circumstance under consideration refers to an external constraint imposed by a human agent or a constraint arising from other sources may constitute a significant consideration. It is true that the distinction is not always clear in borderline cases. But the distinction seems to be clear sufficiently often to be the basis of the notion of negative freedom as the absence of external, humanly imposed constraints that may make an option unavailable to an agent.

It is this conception of negative freedom which is analyzed by van Hees [145]. van Hees [145] makes a distinction between the set of all alternatives which are technically feasible for an individual and the set of all alternatives which are technically feasible and are not ruled out by external and humanly imposed constraints. The former set is called the feasible set by van Hees [145]. The latter set which, by definition, is a subset of the former, is called the opportunity set but, to avoid confusion with the notion of an opportunity set as conceived in the literature, we shall call it the set of permitted options and we shall call its elements permitted options. Note that an option, such as travel abroad, may be permitted in the sense of not being ruled by any human agency and, yet, may be unavailable to the agent because of its limited resources. Let $\mathcal{Y}$ be the set of all ordered pairs (A, B) such that $A, B \in \mathcal{X}$ and $B \subseteq A$. A is to be interpreted as the technically feasible set and B as the set of permitted options. Each ordered pair in $\mathcal{Y}$ is what van Hees [145] calls an opportunity situation. van Hees [145] provides axiomatic characterizations of three cardinality-based rules for ranking opportunity situations, interpreted as rankings in terms of the negative freedom these situations offer. The three rules are defined as follows. For all $(A, B), (A', B') \in \mathcal{Y}$,

$$(A, B) \succeq_{NF}^{1} (A', B') \Leftrightarrow |B| \geq |B'|,$$

$$(A, B) \succeq_{NF}^{2} (A', B') \Leftrightarrow |A' \setminus B'| \geq |A \setminus B|,$$

and

$$(A, B) \succeq_{NF}^{3} (A', B') \Leftrightarrow \frac{|B|}{|A|} \geq \frac{|B'|}{|A'|}.$$

Basu [17] first pointed out a serious limitation of the conventional notion of an opportunity set in a context where outcomes are determined by the actions of several individuals in a group (see also Pattanaik [112]). Consider a two-person, two-commodity exchange economy. Suppose we have a competitive equilibrium, and we want to determine the opportunity set of a consumer in that equilibrium. It is tempting to say that a consumer's opportunity set is its budget set at the equilibrium price vector. However, this seems inappropriate if we are to interpret the opportunity set as the set of all (mutually exclusive) options which are available to the agent and any of which the the agent can choose if it wishes to do so. Suppose, in the competitive equilibrium, consumer 1 chooses the consumption bundle x. Now, if consumer 1 seeks to choose from its budget set a bundle $y \neq x$ then, unless consumer 2 simultaneously changes its choice suitably, the relative prices may change and the desired bundle may not even figure in consumer 1's new budget set. Of course, the problem would not arise if there was a large number of consumers so that no single consumer's choice would affect relative prices. But, in general, such problems in identifying the opportunity set of an individual cannot be ruled out: they would arise in every situation where the outcome for any individual depends on the actions taken by other individuals, as well as the action of the individual under consideration. This then raises the question of how to define the opportunity set of a player participating in a given game. Furthermore, given two different situations represented by two different games, how do we rank the two situations in terms of the freedom of a person who figures as a player in both these games? These are questions that arise naturally from the fundamental concerns of Basu [17], but we do not know of any published work that addresses these conceptual issues relating to the very notion of an opportunity set.

4.3 Well-Being

In the previous two subsections, two alternative approaches to the ranking of opportunity sets were presented. Each of those two approaches is based on a single criterion: the indirect-utility ranking reflects the quality of the final choice, and the cardinality-based ordering is interpreted as a criterion to assess the freedom of choice associated with an opportunity set. In this subsection, we discuss rankings of opportunity sets that make use of several criteria and are interpreted as rankings reflecting several types of characteristics that may be of relevance to the agent.

First, we examine rankings that combine the indirect-utility criterion with considerations of freedom of choice. The interpretation of the use of such rankings is in terms of the overall well-being of an agent: the agent may care about the quality of the final choice as well as the freedom of choice associated with a set of options. As mentioned in Subsection 4.1, this interpretation can be justified both in a welfarist and in a non-welfarist framework.

Natural possibilities to combine considerations of indirect utility and freedom of choice consist in using lexicographic combinations of these criteria. This is done in Bossert, Pattanaik, and Xu [31] for the case where freedom of choice is assessed in terms of the cardinality-based ordering. The indirect-utility-first lexicographic relation $\succeq_U^L$ on $\mathcal{X}$ is defined by letting, for all $A, B \in \mathcal{X}$,

$$A \succeq_U^L B \Leftrightarrow (A \succ_U B \text{ or } [A \sim_U B \text{ and } A \succeq_C B]).$$

This ordering gives priority to indirect utility and only uses freedom of choice (as reflected by the cardinality criterion) as a tie-breaker. Conversely, the following cardinality-first lexicographic ordering $\succeq_C^L$ uses the notion of freedom of choice as its primary criterion and applies the indirect-utility criterion only in cases where the cardinality criterion does not lead to a strict preference between two opportunity sets. The cardinality-first lexicographic relation $\succeq_C^L$ is defined by letting, for all $A, B \in \mathcal{X}$,

$$A \succeq_C^L B \Leftrightarrow (A \succ_C B \text{ or } [A \sim_C B \text{ and } A \succeq_U B]).$$

Among other results, Bossert, Pattanaik, and Xu [31] provide characterizations of those two lexicographic rules. In addition to simple expansion monotonicity, the following axioms are used in those axiomatizations.

Weak Extension Rule: For all $x, y \in X$,

$$xPy \Rightarrow \{x\} \succ \{y\}.$$

Best-Element-Conditional Independence: For all $A, B \in \mathcal{X}$, for all $x \in X \setminus (A \cup B)$ such that $\max(A)Px$ and $\max(B)Px$,

$$A \succeq B \Leftrightarrow A \cup \{x\} \succeq B \cup \{x\}.$$

Simple Indirect Indifference Principle: For all $x, y, z \in X$,

$$xPyPz \Rightarrow \{x, y\} \sim \{x, z\}.$$

Indirect Preference Principle: For all $A \in \mathcal{X}$ such that $|A| > 1$,

$$\{\max(A)\} \succ A \setminus \{\max(A)\}.$$

Simple Priority of Freedom: For all $x, y, z \in X$,

$$xPyPz \Rightarrow \{y, z\} \succ \{x\}.$$

Weak extension rule is an obvious weakening of the extension-rule axiom. Best-element-conditional independence weakens strong independence by restricting the scope of the axiom to situations where the alternative to be added is not a best element in either of the two sets to be compared. The simple indirect indifference principle says that the indifference relation corresponding to the indirect-utility ranking should be respected in comparisons involving two-element sets only. Whereas these three axioms are satisfied by both $\succeq_U^L$ and $\succeq_C^L$, this is not the case for the last two axioms in the above list. The indirect preference principle assigns some priority to best elements, whereas simple priority of freedom says that having two alternatives is better than having no choice at all, even if the object constituting the singleton represeting a no-choice situation is preferred to the two objects in the two-element opportunity set according to the ranking on X. Clearly, $\succeq_U^L$ satisfies the indirect preference principle and violates simple priority of freedom, whereas the reverse is true for $\succeq_C^L$. Note that the indirect preference principle implies weak extension rule, and weak extension rule, best-element-conditional independence, the simple indirect indifference principle, and simple priority of freedom together imply simple expansion monotonicity; see Lemmas 3.2 and 3.3 of Bossert, Pattanaik, and Xu [31].

First, we present a characterization of $\succeq_U^L$ due to Bossert, Pattanaik, and Xu [31]. We provide a self-contained proof here; the proof in Bossert, Pattanaik, and Xu [31] uses more general auxiliary results that are also used in other characterizations.

Theorem 17 *Suppose $\succeq$ is a quasi-ordering on $\mathcal{X}$. $\succeq$ satisfies simple expansion monotonicity, best-element-conditional independence, the simple indirect indifference principle, and the indirect preference principle if and only if $\succeq = \succeq_U^L$.*

Proof. That $\succeq_U^L$ satisfies the required axioms is straightforward to verify. Conversely, suppose $\succeq$ satisfies the axioms. Because $\succeq_U^L$ is an ordering, it is sufficient to prove that, for all $A, B \in \mathcal{X}$,

$$A \sim_U^L B \quad \Rightarrow \quad A \sim B; \tag{17.33}$$
$$A \succ_U^L B \quad \Rightarrow \quad A \succ B. \tag{17.34}$$

Fist, we prove (17.33). Suppose $A \sim_U^L B$ for some $A, B \in \mathcal{X}$. By definition, this is equivalent to $|A| = |B|$ and $\max(A) = \max(B)$. If $A = B$, $A \sim B$ follows from reflexivity. Now suppose $A \neq B$. Let $x = \max(A) = \max(B)$, $C = (A \cap B) \setminus \{x\}$, $D = A \setminus C$, and $E = B \setminus C$. By definition, $D \cap E = \{x\}$ and $\max(D) = \max(E) = x$. Repeated application of the simple indirect indifference principle and best-element-conditional independence yields $D \sim E$. If $A = D$ and $B = E$, we are done; if not, applying best-element-conditional independence as many times as required yields $D \cup C \sim E \cup C$ and hence $A \sim B$.

To prove (17.34), we can distinguish two cases.

(i) $\max(A) = \max(B)$ and $|A| > |B|$;

(ii) $\max(A) P \max(B)$.

In case (i), let $x = \max(A) = \max(B)$ and $y \in A \setminus B$. Let $C \in \mathcal{X}$ be such that $|C| = |A| - |B| > 0$, $C \cap B = \emptyset$, $y \in C$, and xPz for all $z \in C$. By simple expansion monotonicity, $\{x, y\} \succ \{x\}$. If $y \neq \max(C)$, best-element-conditional independence implies $\{x, y, \max(C)\} \succ \{x, \max(C)\}$, and the simple indirect indifference principle implies $\{x, y\} \sim \{x, \max(C)\}$. Because $\succeq$ is transitive, it follows that $\{x, y, \max(C)\} \succ \{x\}$. Using this argument repeatedly, it follows that $C \cup \{x\} \succ \{x\}$. Letting $D = B \setminus \{x\}$, repeated application of best-element-conditional independence yields $C \cup D \cup \{x\} \succ B$. By (17.33), $A \sim C \cup D \cup \{x\}$, and transitivity implies $A \succ B$.

In case (ii), suppose first that $|A| \leq |B|$. Let $r = |B| - |A| + 1$, and let $C \subseteq B$ be the set consisting of the r top-ranked elements in B according to R. By the indirect preference principle, $\{\max(A)\} \succ C$, and repeated application of best-element-conditional independence yields $\{\max(A)\} \cup B \setminus C \succ B$. By (17.33), $A \sim \{\max(A)\} \cup B \setminus C$ and hence $A \succ B$ by transitivity.

If $|A| > |B|$, $A \succ B$ follows from combining the result of case (i), the above observation for $|A| \leq |B|$, and transitivity. ■

Analogously, we obtain the following characterization of $\succeq_C^L$.

Theorem 18 *Suppose $\succeq$ is a quasi-ordering on $\mathcal{X}$. $\succeq$ satisfies weak extension rule, best-element-conditional independence, the simple indirect indifference principle, and simple priority of freedom if and only if $\succeq = \succeq_C^L$.*

Proof. See Bossert, Pattanaik, and Xu [31]. ■

The rankings characterized in Theorems 17 and 18 are complete (although this property does not have to be assumed explicitly—it follows as a consequence of the axioms in the theorem statements). An example for an incomplete ranking of opportunity sets based on considerations of indirect utility and freedom of choice is a dominance relation which ranks opportunity sets by declaring one set to be at least as good as another if and only if the best element of the former is at least as good as the best element of the latter and the first set has at least as many elements as the second. Formally, we define the indirect-utility-cardinality dominance relation $\succeq_{UC}^{D}$ by letting, for all $A, B \in \mathcal{X}$,

$$A \succeq_{UC}^{D} B \Leftrightarrow (A \succeq_U B \text{ and } A \succeq_C B).$$

See Bossert, Pattanaik, and Xu [31] for a more detailed discussion and a characterization.

An ordering that is related to $\succeq_U^L$ in the sense that it assigns priority to considerations of indirect utility is the indirect-utility leximax ordering $\succeq_{\max}^U$ introduced in Bossert, Pattanaik, and Xu [31]. As is the case for $\succeq_U^L$, the primary criterion for a set comparison is the best element. If the best elements of the two sets to be compared are the same, the next criterion is given by comparing the second-best elements of those sets, and the procedure continues until either the two sets are idential and declared indifferent, or a strict preference is established before exhausting at least one of the sets, or one of the sets is exhausted and the other is not, in which case the set with more elements is declared better. Formally, we let, for all $A, B \in \mathcal{X}$,

$$\begin{aligned} A \succeq_{\max}^U B \quad \Leftrightarrow \quad & (A = B \text{ or} \\ & [|A| > |B| \text{ and } a_{\rho_A(i)} = b_{\rho_B(i)} \text{ for all } i \in \{1, \ldots, |A|\}] \text{ or} \\ & [\exists i \in \{1, \ldots, \min\{|A|, |B|\}\} \text{ such that} \\ & a_{\rho_A(j)} = b_{\rho_B(j)} \text{ for all } j < i \text{ and } a_{\rho_A(i)} P b_{\rho_B(i)}]) \end{aligned}$$

(recall that, as defined in Section 3, $\rho_A \colon \{1, \ldots, |A|\} \to \{1, \ldots, |A|\}$ is a permutation of $\{1, \ldots, |A|\}$ such that $a_{\rho_A(i)} R a_{\rho_A(i+1)}$ for all $A \in \mathcal{X}$ and for all $i \in \{1, \ldots, |A| - 1\}$).

As opposed to $\succeq_U^L$, the ordering $\succeq_{\max}^U$ does not explicitly take into account the cardinalities of the sets to be compared—the number of elements can only matter to the extent that any set with at least two elements is considered better than its strict subsets. This behaviour of the ranking when one set becomes empty but the other does not in the lexicographic procedure also distinguishes this ordering from the leximax ordering $\succeq_{\max}^L$ introduced in Section 3: if one of the sets becomes empty in one of the above-described steps, the smaller rather than the larger set is declared better according to $\succeq_{\max}^L$. A characterization of $\succeq_{\max}^U$ can be found in Bossert, Pattanaik, and Xu [31].

The two well-being orderings $\succeq_U^L$ and $\succeq_C^L$ axiomatized in Bossert, Pattanaik, and Xu [31] can be thought of as specific results of aggregating the cardinality-based freedom ranking and the indirect-utility ranking. However, if one thinks of $\succeq_U^L$ and $\succeq_C^L$ in that fashion, then one can view the aggregation procedure as a social choice rule that takes individual preferences and aggregates them so as to arrive at social preferences, the freedom ranking and the indirect-utility ranking being the 'individual' orderings and the well-being ranking the 'social' ordering. However, unlike the standard Arrovian framework that admits different profiles of individual orderings, here we have only a fixed profile of two orderings. Therefore, the formal analogy with the social choice problem only holds with respect to social choice in the single-profile framework, as distinct from the multi-profile framework of Arrow [5]; for a discussion of the distinction between single-profile and multi-profile social choice see, for example, Sen [134]. It is this formal analogy which is exploited in Dutta and Sen [45] who work with a finite set X. Dutta and Sen [45], however, do not proceed simply by starting with the cardinality-based freedom ranking and the indirect-utility ranking and then aggregating these two orderings into an overall well-being ordering—indeed, these two orderings do not figure as primitive notions in their formal analysis. What makes their approach interesting and general is that they start with axioms regarding the ranking of opportunity sets (some of which are taken from Bossert, Pattanaik, and Xu [31]) and make use of well-known results in the theory of social choice to provide a joint characterization of the two orderings $\succeq_U^L$ and $\succeq_C^L$. Pursuing the analogy with the social choice problem, note that, when the well-being ranking must be equal to $\succeq_U^L$ or to $\succeq_C^L$, this means that one of those two orderings must have 'dictatorial' priority over the other in the aggregation process. Dutta and Sen [45] also provide an axiomatic characterization of each of the two rankings $\succeq_U^L$ and $\succeq_C^L$ taken by itself, the axiomatizations being different from those provided by Bossert, Pattanaik, and Xu [31]. Lastly, they provide a characterization of a new class of rankings $\succeq_\lambda$, where $\lambda = (\lambda_1, \lambda_2)$ and $\lambda_1, \lambda_2 > 0$. Given λ, $\succeq_\lambda$ is defined by letting, for all $A, B \in \mathcal{X}$,

$$A \succeq_\lambda B \Leftrightarrow \lambda_1 |\{x \in X \mid \max(A) R x\}| + \lambda_2 |A| \geq \lambda_1 |\{x \in X \mid \max(B) R x\}| + \lambda_2 |B|.$$

Note that, unlike $\succeq_U^L$ and $\succeq_C^L$, the orderings $\succeq_\lambda$ allow for a trade-off between the consideration of freedom and the consideration of indirect utility.

Another approach to combining considerations of freedom of choice and indirect utility can be found in Gravel [66, 67]. Gravel uses an extended framework where not only opportunity sets in $\mathcal{X}$ but pairs (a, A) with $A \in \mathcal{X}$ and $a \in A$ are to be ranked. The interpretation is that a is chosen from the set of feasible options A, and a ranking capable of comparing those choice situations is to be established. Gravel [67] provides an impossibility result showing that, given a certain interpretation of the notion of freedom of choice and some additional properties, specific axioms combining considerations of freedom and indirect utility are incompatible; see also Gravel [66] for a discussion. However, his impossibility result crucially hinges on the assumption that a set A offers more freedom than a set B only if B is a proper subset of A. Because this assumption is highly restrictive and can be considered questionable, the resulting impossibility theorem does not consititute as serious a difficulty as it may appear at first glance. Further impossibility results regarding preference-based rankings of opportunity sets can be found in Puppe [119].

Finally, another contribution concerned with set rankings in terms of well-being is Bossert [26]. In that paper, opportunity set ranking rules that assign a ranking $\succeq$ on $\mathcal{X}$ to any ordering R on X (as opposed to a single fixed ordering R) are considered. The main result there is a characterization of rankings with an additively separable structure. If more than ordinal information regarding the quality of the alternatives in X is assumed to be available, it is illustrated in Bossert [26] how information invariance conditions can be used to identify subclasses of those additively separable rankings. See Bossert [26] for details of this alternative model.

4.4 Flexibility and Consequences

The term flexibility appeals to a multi-period decision situation, where decisions at period t leave a number of possibilities open for period $t+1$, while closing others. A decision maker has larger flexibility than another at t if its previous actions have left it with a wider set of open courses of action. Agents who are certain of what course of action will be best for them, under any circumstances, need not find any advantage in having any flexibility: they will treat any two sequences of decisions alike, as long as they allow for the same optimal choices. In other terms, agents who are sure of what they will want to choose, and who know exactly what decisions will be available to them at each period, will rank sequences of actions by their indirect utility alone. Preferences for flexibility will arise in the presence of some uncertainty about the value and the availability of future courses of action. Flexibility is the possibility to adapt to contingencies.

Koopmans [84] discusses preferences for flexibility in general terms, pointing out the complexity of the notion when several time periods and different types of decisions are involved. He also credits Hart [70] for early discussions of the topic. Goldman [64, 65] and Jones and Ostroy [76] use the notion of flexibility as a theoretical foundation for the idea that liquidity, one of the services provided by money, is a source of money's utility. Kreps [89] refers to these antecedents and provides an axiomatic foundation for the notion of flexibility.

Kreps' [89] formulation does not insist on the multi-period possibilities pointed out by Koopmans, and focuses on a two-period formulation, where today's decisions narrow down the possibilities left for the decision maker in the following (and last) decision round. This links his analysis to the type of models that our survey is restricted to, namely those where the objects to be ranked are sets. In what follows, we shall assume that an agent is facing a choice today that will narrow down its set of possible choices for tomorrow. Moreover, we assume that the agent only derives satisfaction from the final future choice. Hence, today's choice is instrumental; it is the means by which the agent determines what opportunities will still be available when the time for the final decision comes (Kreps' [89] Section 4 extends the analysis to the case where today's choices also generate utility). Under these circumstances, today's actions can be identified with the sets of opportunities that they leave open. Ranking today's actions is equivalent to ranking the sets of opportunities that they leave for tomorrow.

A ranking of sets exhibits preference for flexibility if every set is strictly preferred to any of its proper subsets, that is, if the agent always considers closing down an opportunity to be a loss. This requirement is monotonicity with respect to set inclusion as defined in Subsection 4.2.

The generic appeal to uncertainty when justifying the preferences of agents for larger sets of future opportunities can now be subjected to some test. Can we provide a formal model of a rational agent whose ranking of oppotunity sets satisfies monotonicity with respect to set inclusion, while being exclusively concerned with the utility of its final choices? If so, does our model imply any additional restrictions on the way that these agents would rank sets, other than monotonicity with respect to set inclusion?

The model suggested by Kreps [89] is the following. When making decisions in the first stage, the agent is uncertain about the preference relation that will govern its choice of a final alternative in the second stage. One can represent the agent's uncertainty over future preferences by assuming that the agent is endowed with a state-dependent utility function, and that this uncertainty will be resolved by the second period. The agent will then choose its utility-maximizing alternative according to its second-stage preferences, among those alternatives not excluded by its first-period decision. Under this interpretation, an agent would rank actions, and thus the sets of choices they leave open for the second stage, in terms of the expected utility of their second-stage choices, with the expectation relative to the probabilities attached to each state.

To introduce this idea formally, let $\mathbb{R}$ ($\mathbb{R}_+$) be the set of all (all nonnegative) real numbers, and let S be a finite, nonempty set of states of the world. A lottery on S is a function $\bar{\ell}\colon S \to \mathbb{R}_+$ such that $\sum_{s \in S} \bar{\ell}(s) = 1$. The set of all lotteries on S is denoted by $\bar{\mathcal{L}}$. A state-dependent utility function is a function $\bar{u}\colon X \times S \to \mathbb{R}$.

Kreps [89] provides necessary and sufficient conditions for an ordering over sets to be representable in terms of such a model. Because he considers orderings on $\mathcal{X}$ that are not necessarily linear, an axiom in addition to monotonicity with respect to set inclusion is required. It is an independence condition the scope of which is restricted to situations of indifference.

Indifference Independence: For all $A, B, C \in \mathcal{X}$,

$$A \sim A \cup B \Rightarrow A \cup C \sim A \cup B \cup C.$$

Kreps' [89] representation theorem can now be stated as follows.

Theorem 19 *Suppose $\succeq$ is an ordering on $\mathcal{X}$. $\succeq$ satisfies monotonicity with respect to set inclusion and indifference independence if and only if there exist a nonempty and finite set of states S, a state-dependent utility function $\bar{u}\colon X \times S \to \mathbb{R}$, and a lottery $\bar{\ell} \in \bar{\mathcal{L}}$ such that, for all $A, B \in \mathcal{X}$,*

$$A \succeq B \Leftrightarrow \sum_{s \in S} \bar{\ell}(s) \max_{x \in A} \bar{u}(x, S) \geq \sum_{s \in S} \bar{\ell}(s) \max_{x \in B} \bar{u}(x, S).$$

Proof. See Kreps [89]. ■

Note that if $\succeq$ is assumed to be linear (which we will do to simplify exposition in what follows), indifference independence is vacuously satisfied, and the essential message of Kreps' [89] representation theorem stated above is that, indeed, any linear ordering of sets satisfying monotonicity with respect to set inclusion can be generated as the result of uncertainty over future preferences, and that no further restrictions are imposed by this model on the admissible rankings of sets.

The restriction imposed by monotonicity with respect to set inclusion is so weak that many other representations (not necessarily one based on expected utility) are also possible. Specifically, the agent's way to combine the maximal utilities at each state into a utility index for sets could assume different forms. Hence, some authors have sought to enrich the model and investigate conditions under which the expected-utility formulation becomes compelling. See, for example, Dekel, Lipman, and Rustichini [40] or Nehring [99].

We now want to suggest an alternative source of preference for flexibility, one that does not require agents to be uncertain about their future preferences. This is already mentioned in Koopmans [84] but not pursued. Following Barberà and Grodal [13], we may think of agents who are aware that their first-stage choices exclude some options for the future, but are not certain that all non-excluded options will actually be available in the second stage. Sets can be viewed as lists of options that are not definitely excluded. But the agent may be uncertain as to what part of a non-excluded set will still be available for a final choice. To motivate this scenario, consider, for example, actions with an environmental impact. Some alternatives may become definitely impossible after a first action is undertaken, while others may still be open for future choice. Yet, additional events like a natural disaster can further narrow down the set of options. As a less dramatic example, consider the choice of a restaurant. Some meals that are never available at that restaurant are definitely excluded by making a reservation there. But some of the items which are not initially excluded, because they are on the restaurant's menu, may be unavailable on the day the agent actually goes to the restaurant.

Keeping this interpretation in mind, it is natural to assume that an agent will be endowed with a von Neumann – Morgenstern utility function (see von Neumann and Morgenstern [147]) and lotteries indicating, for each subset of X, the probability that the agent attaches to the survival of a given subset of A if A is the set of options not excluded by the first-stage action. When evaluating an action that excludes all elements not in A, and thus leaves A as possible options, the agent can perform an expected-utility calculation. It can compute the expected value of its best choice out of each subset of A, given the probability it attaches to the event that only alternatives in that subset will be available. This induces a ranking of actions and, thus, of sets of (non-excluded) alternatives.

We now express this model formally and present a simple constructive proof that a strict version of monotonicity with respect to set inclusion is the only restriction imposed by the model, provided that $\succeq$ is a linear ordering. Furthermore, it is convenient to include the empty set as one of the sets to be ranked, that is, we want to establish a linear ordering on $\mathcal{X}_0$.

The above-mentioned strict version of monotonicity with respect to set inclusion is defined as follows.

Strict Monotonicity with Respect to Set Inclusion: For all $A, B \in \mathcal{X}_0$,

$$B \subset A \Rightarrow A \succ B.$$

For a linear ordering $\succeq$ on $\mathcal{X}_0$ and $A \in \mathcal{X}_0$, let $r(\succeq, A) = |\{B \in \mathcal{X} \mid B \succ A\}|$ be the rank of A according to $\succeq$. Clearly, $r(\succeq, A) = 0$ if and only if A is the best element in $\mathcal{X}_0$ according to $\succeq$. Let $\mathcal{L}$ be the set of all lotteries on $\mathcal{X}_0$, that is, $\ell \in \mathcal{L}$ if and only if $\ell\colon \mathcal{X}_0 \to \mathrm{I\!R}_+$ is such that $\sum_{A \in \mathcal{X}_0} \ell(A) = 1$. For $\ell \in \mathcal{L}$ and $A \in \mathcal{X}_0$, define the function $\ell_A\colon \mathcal{X}_0 \to \mathrm{I\!R}_+$ by letting $\ell_A(B) = \sum_{C \in \mathcal{X}_0 | C \cap A = B} \ell(C)$ for all $B \in \mathcal{X}_0$. The interpretation of ℓ_A is as follows. Assume that the probabilities of the elements in $\mathcal{X}_0$ are given by ℓ and that the agent has chosen the set A in the first stage. Then the probability of the event that the agent has to choose an element in the set B is given by $\ell(B)$. The following theorem is an immediate consequence of the definition of ℓ_A. We use the notation $\mathcal{X}_{-A}$ for the set of all finite subsets of $X \setminus A$, where $A \in \mathcal{X}_0$.

Theorem 20 *Suppose $\ell \in \mathcal{L}$. For all $A \in \mathcal{X}_0$,*

(a) $\ell_A \in \mathcal{L}$;
(b) for all $B \in \mathcal{X}$ such that $B \not\subseteq A$, $\ell_A(B) = 0$;
(c) $\ell_A(\emptyset) = \sum_{B \in \mathcal{X}_{-A}} \ell(B)$.

Proof. Follows immediately from the definition of ℓ_A. ■

Furthermore, we obtain

Theorem 21 *Suppose $\ell \in \mathcal{L}$. For all $A, B \in \mathcal{X}_0$ such that $A \subseteq B$,*

$$\ell_A(C) = \sum_{E \in \mathcal{X}_0 | E \cap A = C} \ell_B(E)$$

for all $C \in \mathcal{X}_0$.

Proof. By definition, $\ell_A(C) = \sum_{D \in \mathcal{X}_0 | D \cap A = C} \ell(D)$. For each $D \in \mathcal{X}_0$ such that $D \cap A = C$, there exists a unique $E \in \mathcal{X}_0$ such that $D \cap B = E$ and $E \cap A = C$. Hence,

$$\sum_{D \in \mathcal{X}_0 | D \cap A = C} \ell(D) = \sum_{E \in \mathcal{X}_0 | E \cap A = C} \sum_{D \in \mathcal{X}_0 | D \cap B = E} \ell(D) = \sum_{E \in \mathcal{X}_0 | E \cap A = C} \ell_B(E),$$

which proves the claim. ■

Let $X_0 = X \cup \{\emptyset\}$. We consider utility functions $U\colon X_0 \to \mathbb{R}$ such that $U(\emptyset) = 0$. For a given utility function u and a lottery $\ell \in \mathcal{L}$, the expected-opportunity function $V\colon \mathcal{X}_0 \to \mathbb{R}$ is defined by letting, for all $A \in \mathcal{X}_0$, $V(A) = \sum_{B \in \mathcal{X}_0} \max_{x \in B \cup \{\emptyset\}} \ell_A(B) U(x)$. The value of the function V at A computes the expected utility for the agent if it selects the set A in the first stage. Thus, we assume that if B is the set of surviving alternatives in A, the agent will choose the best alternative in B. Yet, there also is the possibility of choosing $\emptyset$ in case all the alternatives in B are worse than $\emptyset$. Hence, $\max_{x \in B \cup \{\emptyset\}} U(x)$ is the utility the agent obtains if the alternatives in $C \in \mathcal{X}_0$ survive and $C \cap A = B$.

The linear ordering $\succeq$ is an expected-opportunity ranking if and only if there exist a lottery $\ell \in \mathcal{L}$ and a utility function $U\colon X_0 \to \mathbb{R}$ such that, for all $A, B \in \mathcal{X}_0$,

$$A \succ B \Leftrightarrow V(A) > V(B)$$

where V is the expected-opportunity function corresponding to ℓ and U. The following theorem presents some consequences of the above definitions for utility functions and lotteries underlying an expected-opportunity ranking.

Theorem 22 *Suppose $\succeq$ is an expected-opportunity ranking for $\ell \in \mathcal{L}$ and $U\colon X_0 \to \mathbb{R}$ is a utility function such that $U(\emptyset) = 0$.*

(a) $V(A) \geq 0$ for all $A \in \mathcal{X}_0$;
(b) for all $x \in X$, $U(x) > 0$ and there exists $A \in \mathcal{X}_0$ such that $x \in A$ and $\ell(A) > 0$.

Proof. (a) follows immediately because, by assumption, $U(\emptyset) = 0$ and thus $\max_{x \in B \cup \{\emptyset\}} \ell_A(B) U(x) \geq 0$ for all $A \in \mathcal{X}_0$. To prove (b), note first that $V(\emptyset) = 0$ by definition. Hence, since $\succeq$ is linear, $V(A) > 0$ for all $A \in \mathcal{X}$. Consequently, for all $x \in X$,

$$V(\{x\}) = \sum_{B \in \mathcal{X}_0} \max_{y \in B \cup \{\emptyset\}} \ell_{\{x\}}(B) U(y) = \max(0, U(x)) \sum_{x \in A} \ell(A) > 0$$

for some $A \in \mathcal{X}_0$ such that $x \in A$, and (b) follows. ■

Analogously to Kreps' [89] result in the linear case, the following theorem, due to Barberà and Grodal [13], states that all restrictions imposed by this model are given by strict monotonicity with respect to set inclusion.

Theorem 23 *Suppose X is finite and $\succeq$ is a linear ordering on $\mathcal{X}_0$. $\succeq$ satisfies strict monotonicity with respect to set inclusion if and only if $\succeq$ is an expected-opportunity ranking.*

Proof. To prove the if-part of the statement, let $\succeq$ be an expected-opportunity ranking and suppose $A, B \in \mathcal{X}_0$ are such that $B \subset A$. Then

$$\begin{aligned} V(A) &= \sum_{C \in \mathcal{X}_0} \max_{x \in C \cup \{\emptyset\}} \ell_A(C) U(x) \\ &= \sum_{D \subseteq B} \sum_{D \in \mathcal{X}_0 | D \cap B = C} \max_{x \in C \cup \{\emptyset\}} \ell_A(C) U(x) \\ &\geq \sum_{D \subseteq B} \sum_{D \in \mathcal{X}_0 | D \cap B = C} \max_{x \in D \cup \{\emptyset\}} \ell_A(C) U(x) \\ &= \sum_{D \subseteq B} \max_{x \in D \cup \{\emptyset\}} U(x) \sum_{D \in \mathcal{X}_0 | D \cap B = C} \ell_A(C) \\ &= \sum_{D \subseteq B} \max_{x \in D \cup \{\emptyset\}} \ell_B(D) U(x) = V(B) \end{aligned}$$

where the next-to-last equality follows from Theorem 21. Since we know that $V(A) \neq V(B)$ (because $\succeq$ is linear and an expected-opportunity ranking), it follows that $V(A) > V(B)$ and hence $A \succ B$.

We now prove the only-if part of the theorem by showing that a linear ordering $\succeq$ satisfying strict monotonicity with respect to set inclusion is an expected-opportunity ranking corresponding to some lottery $\ell \in \mathcal{L}$ and the utility function U defined by $U(\emptyset) = 0$ and $U(x) = 1$ for all $x \in X$. First, note that the expected-opportunity function V corresponding to ℓ and U is given by $V(A) = 1 - \ell_A(\emptyset)$ for all $A \in \mathcal{X}_0$. Hence, V represents $\succeq$ if and only if, for all $A, B \in \mathcal{X}_0$,

$$A \succ B \Leftrightarrow \ell_A(\emptyset) < \ell_B(\emptyset). \tag{17.35}$$

Thus, we have to construct a lottery ℓ satisfying (17.35).

Define the complement $\succeq^*$ of $\succeq$ by letting, for all $A, B \in \mathcal{X}_0$,

$$A \succeq^* B \Leftrightarrow X \setminus A \succeq X \setminus B.$$

We now define two functions $s: \mathcal{X}_0 \to \mathbb{R}$ and $t: \mathcal{X}_0 \to \mathbb{R}$ recursively over the rank of the sets according to $\succeq^*$. Because $\succeq$ satisfies strict monotonicity with respect to set inclusion, $r(\succeq^*, \emptyset) = 0$. Let $s(\emptyset) = t(\emptyset) = 0$. Now let $A \in \mathcal{X}$ be the unique set such that $r(\succeq^*, A) = 1$. Because $\succeq$ satisfies strict monotonicity with respect to set inclusion, there exists $x \in X$ such that $A = \{x\}$. Define $s(A) = r(\succeq^*, A) = 1$ and $t(A) = s(A) - \sum_{B \in \mathcal{X}_0 | A \succ^* B} s(B) = 1 - 0 = 1$.

Now assume the values of s and t have been defined for all $C \in \mathcal{X}_0$ such that $r(\succeq^*, C) \leq k$ with $k \geq 1$. Let $A, B \in \mathcal{X}_0$ be such that $r(\succeq^*, A) = k$ and $r(\succeq^*, B) = k + 1$. By strict monotonicity with respect to set inclusion, the values of s and t are defined for all sets $C \subset B$. We now define

$$s(B) = \max(s(A) + 1, \sum_{C \subset B} t(C))$$

and

$$t(B) = s(B) - \sum_{C \subset B} t(C).$$

Because X (and, thus, $\mathcal{X}_0$) is finite, s and t are well-defined. By definition of s,

$$s(A) > s(B) \Leftrightarrow B \succ^* A \Leftrightarrow X \setminus B \succ X \setminus A$$

and hence

$$A \succ B \Leftrightarrow s(X \setminus B) > s(X \setminus A) \tag{17.36}$$

for all $A, B \in \mathcal{X}_0$. Furthermore, by definition of s and t,

$$s(A) = \sum_{B \subseteq A} t(B) \tag{17.37}$$

for all $A \in \mathcal{X}_0$.

Now define the lottery $\ell \in \mathcal{L}$ by letting $\ell(A) = t(A)/T$ for all $A \in \mathcal{X}_0$, where $T = \sum_{B \in \mathcal{X}_0} t(B)$. By Theorem 20 and the definition of ℓ,

$$\ell_A(\emptyset) = \sum_{B \in \mathcal{X}_{-A}} \ell(B) = \sum_{B \in \mathcal{X}_{-A}} t(B)/T.$$

Thus, by (17.37), $\ell_A(\emptyset) = s(X \setminus A)/T$. Combined with (17.36), it follows that (17.35) is satisfied for ℓ. ■

Clearly, the utility function U and the lottery ℓ in Theorem 23 are not unique. For example, for an arbitrary linear ordering R on X, we can find a utility function $\bar{U}: X_0 \to \mathbb{R}$ and a lottery $\ell \in \mathcal{L}$ such that the resulting expected-opportunity function represents $\succeq$ and, moreover, the restriction of $\bar{U}$ to X represents R.

We can now explain why Kreps' [89] analysis already contained everything we can learn from the above representation result of Barberà and Grodal [13]. Kreps' definitions allow for any choice of the set of states and for any utility function defined on this set. In his proof, the states are sets of alternatives and the utilites are of the type exhibited above. To illustrate this observation, assume that V is an expected-opportunity function for the expected-opportunity ranking $\succeq$ with a lottery ℓ and a utility function U. By definition of a utility function, $U(\emptyset) = 0$ and, from Theorem 22, $U(x) > 0$ for all $x \in X$. Hence,

$$\begin{aligned} V(A) &= \sum_{B \in \mathcal{X}_0} \max_{x \in B \cup \{\emptyset\}} \ell_A(B) U(x) \\ &= \sum_{\emptyset \neq B \subseteq A} \max_{x \in B} \ell_A(B) U(x) \\ &= \sum_{D \in \mathcal{X} | A \cap D \neq \emptyset} \max_{x \in A \cap D} \ell(D) U(x) \end{aligned}$$

for all $A \in \mathcal{X}_0$.

We shall now reinterpret the expected-opportunity function V in terms of uncertainty about the agent's preferences in the second stage. Let the set of states be $S = \mathcal{X}_0$ and let ℓ be the lottery on S. Define the state-contingent utility function $\bar{u}: X \times \mathcal{X}_0 \to \mathbb{R}$ by letting

$$\bar{u}(x, A) = \begin{cases} U(x) & \text{if } x \in A; \\ 0 & \text{if } x \notin A \end{cases}$$

for all $(x, A) \in X \times \mathcal{X}_0$. Now assume that, in each state $A \in \mathcal{X}_0$, the agent maximizes $\bar{u}(x, A)$ by choice of x. The expected-utility ranking of the subsets is then represented by the function $\bar{V}: \mathcal{X}_0 \to \mathbb{R}$ defined by letting

$$\begin{aligned} \bar{V}(A) &= \sum_{B \in \mathcal{X}_0} \max_{x \in B} \ell(B) \bar{u}(x, B) \\ &= \sum_{B \in \mathcal{X} | A \cap B \neq \emptyset} \max_{x \in A \cap B} \ell(B) \bar{u}(x, B) \\ &= \sum_{B \in \mathcal{X} | A \cap B \neq \emptyset} \max_{x \in A \cap B} \ell(B) U(x) \\ &= V(A) \end{aligned}$$

for all $A \in \mathcal{X}_0$. Hence, the expected-opportunity function V can be reinterpreted as a representation in terms of uncertainty of second-stage preferences. Why, then, examine uncertain opportunities rather than uncertain preferences as sources of flexibility? First of all, because both phenomena are attractive and their consequences worth exploring. Second, because each interpretation may call for the investigation of related issues which are natural under one of them and may not be under the other.

Here is an example. Theorem 23 states that, under the uncertain-opportunities interpretation, a representation exists for any linear ranking of sets satisfying strict monotonicity with respect to set inclusion. But this representation may require the survival probability of a non-excluded alternative to be dependent on the set of alternatives which are not excluded by an action. This may be reasonable in some cases but need not be in others. If we think that the survival probability is an independent characteristic of each alternative, then we may want to restrict attention to those rankings of sets which can be rationalized in the following sense.

We say that $\succeq$ is an expected-opportunity ranking with independent survival probabilities if and only if there exists a utility function $U\colon X_0 \to \mathbb{R}$ with $U(\emptyset) = 0$ and a survival lottery $\hat{\ell}\colon X_0 \to \mathbb{R}_+$ with $\sum_{x \in X_0} \hat{\ell}(x) = 1$ such that, with $\ell(A) = \prod_{x \in A} \hat{\ell}(x) \prod_{y \notin A}(1 - \hat{\ell}(y))$ for all $A \in \mathcal{X}_0$, the resulting expected-opportunity function V represents $\succeq$.

Not all rankings satisfying strict monotonicity with respect to set inclusion are expected-opportunity rankings with independent survival probabilities. For example, let $X = \{x, y, z\}$ and suppose

$$\{x, y, z\} \succ \{z, y\} \succ \{z, x\} \succ \{x, y\} \succ \{x\} \succ \{y\} \succ \{z\}.$$

$\succeq$ satisfies the strict inclusion property. One of the three singletons $\{x\}$, $\{y\}$, and $\{z\}$ must be assigned the maximal utility among the three. To be an expected-opportunity ranking with independent survival probabilities, $\succeq$ has to satisfy the requirement that adding this particular singleton to any two disjoint sets not containing it cannot reverse their relative ranking according to $\succeq$. However, in this example, adding $\{x\}$ to both reverses the relative ranking of $\{y\}$ and $\{z\}$, adding $\{y\}$ to both reverses the relative ranking of $\{x\}$ and $\{z\}$, and adding $\{z\}$ to both reverses the relative ranking of $\{x\}$ and $\{y\}$. Therefore, none of the three can be assigned maximal utility and, thus, $\succeq$ is not an expected-opportunity ranking with independent survival probabilities. Characterizing all expected-opportunity rankings with independent survival probabilities is an unresolved problem; see Barberà and Grodal [13] for a summary of the progress made in that respect so far.

We have already mentioned that, unless further structure is imposed in the model, uncertain states play an instrumental role in the representation of a set ranking. The interpretation of what these states stand for remains a matter of choice at this stage. Kreps' [89] provides an as-if description of the ranking to be represented. However, one may have higher expectations from a theory of delayed actions. In particual, contract theory seems to beg for the distinction between foreseen and unforeseen contingencies, which could then be turned into a foundation for the distinction between complete and incomplete contracts. Dekel, Lipman and Rustichini [41] review different attempts at turning this intuitively attractive distinction into a well-grounded theoretical concept. Some of these attempts have been tied to the notion of flexibility, to the extent that agents may want to retain additional opportunities at hand in the face of an uncertain future. Kreps [90] also comments on these efforts. One recent contribution combining the notion of flexibility with the operationalization of an idea of unforeseen contingencies is that of Nehring [99]. However, Nehring's [99] approach requires the full richness of decision-theoretic models, involving the distinction between acts, states, and consequences, to make its points.

Arlegi and Nieto [3] remark that, if preferences on sets are induced by a desire to postpone choice in the presence of uncertainty over future preferences, then agents may rank certain sets according to the indirect-utility criterion, while respecting the strict version of monotonicity with respect to set inclusion in other pairwise comparisons. Indeed, it is natural to assume that agents may be absolutely certain about some parts of their preferences, while uncertain about others. This may be reflected by attributing to an agent a partial ranking of pairs it is sure how to rank, and allow for any completion of this partial order to stand for its possible preferences in the second stage. Under such a formulation, rankings of sets which always treat sets as better than their subsets, on the one hand, and rankings of sets according to indirect utility, on the other hand, appear as two extreme cases, with partial uncertainty regarding preferences as the general case. Other approaches to the ranking of opportunity sets where there are more than just one ranking of the alternatives themselves that are considered relevant can be found in Arrow [6], Foster [57], and Nehring and Puppe [102]. See also Suppes [144] for a discussion of the relationship between maesuring freedom and measuring uncertainty.

To conclude this subsection, we briefly mention another approach to the ranking of opportunity sets where uncertainty plays a role. In that model, uncertainty comes into play because the menu items offered to a decision maker as possible choices may not coincide with the objects the decision maker ultimately cares about—the consequences of those choices. If this is the case, we may encounter situations where the choice of a menu item does not necessarily lead to a unique outcome but, rather, a set of possible outcomes, and this uncertainty about the consequences of choosing menu items can be expected to influence an agent's ranking of the opportunity sets it may face. This is the interpretation of an example in Sen [139] suggested in Bossert [29]. In that case, a possible criterion for ranking opportunity sets is the amount of information they convey about consequences (what Sen [139] calls the epistemic value of a menu). Possible orderings with such an interpretation are characterized and discussed in Bossert [28] and in Naeve and Naeve-Steinweg [98]; see those papers for more detailed discussions.

5 SETS AS FINAL OUTCOMES

5.1 Conceptual Issues

In this section, we examine the ranking of sets the members of which are not mutually exclusive. As we shall see, theories such as those concerned with matching, voting, and coalition formation make use of such rankings.

Matching, the organization of coalitions to work jointly on some common purpose, the choice of assemblies, and the election of new members to join an organization are examples of special ways to form groups. Since groups of agents or objects may be mutually compatible and singletons are particular instances of more general sets, one can take preferences over sets to be a primitive. Then, since the restriction of these preferences to singletons will be part of that primitive information, one may investigate the relationship between the ranking of singletons and that of larger sets.

Alternatively, one can take the view that the primitive preferences are those over single objects, and that preferences over over sets are the result of aggregating the preferences over the constituent members of these sets in an appropriate manner.

These two positions have been taken by different authors, and we feel that the distinction is often a matter of presentation rather than the expression of opposing views. In this section, we review results coming from both of the above-mentioned strands of literature.

5.2 Fixed-Cardinality Rankings

Consider the college-admissions problem (see, for example, Gale and Shapley [59] and Roth and Sotomayor [128]), which is the standard example of a many-to-one matching problem. Colleges, by admitting some of their applicants, end up with a new class of first-year students. The connection between the college's valuation of each individual student and its valuation of the different possible classes it can admit is determinant in the allocation process. In particular, the existence of stable matchings and the structure of the set of matchings hinge on the type of preferences that colleges may have over sets of students.

Because colleges are often assumed to have a fixed quota $q \in \mathbb{N}$ specifying the maximal number of students they can admit, matching theory frequently concentrates on the preferences of colleges over sets of size q. The result we discuss in this subsection refers to the ranking of sets of fixed size, following the interpretation just provided. Consider a universal set X of students, and let $q \in \mathbb{N}$ be the number of slots that can be filled by a college. In order to remain within the same framework of other sections of this chapter, we assume that R is a linear ordering on X, although the result presented in this subsection can be modified to accommodate some cases where R is not necessarily linear. This may be important because sometimes these fixed-dimensional vectors can also contain, in addition to the names of students, the name of the college in order to represent vacant slots. If the name of the college appears several times, where each occurrence is treated as a different entity, it is appropriate to assume that these empty slots are indifferent to each other. The consequences of admitting indifference would be that the following restriction has to be strengthened appropriately, and the statement of the characterization result presented in this context has to be amended accordingly. X may be finite or infinite, but it must contain at least $2q+1$ elements in order for the main result presented here to be true. Let $\succeq_q$ be a relation on $\mathcal{X}_q$.

A responsiveness axiom imposed on college preferences is used, for example, in Roth's [127] analysis of the college admissions problem. This condition requires that if, ceteris paribus, one element x in a set $A \in \mathcal{X}_q$ is replaced by another element $y \in X \setminus A$, then the relative ranking of the new set and the original set according to $\succeq_q$ is determined by the relative ranking of y and x according to R.

Responsiveness: For all $A \in \mathcal{X}_q$, for all $x \in A$, for all $y \in X \setminus A$,

$$[A \succeq_q (A \setminus \{x\}) \cup \{y\} \Leftrightarrow xRy] \text{ and } [(A \setminus \{x\}) \cup \{y\} \succeq_q A \Leftrightarrow yRx].$$

The above axiom, which reflects a basic lack of complementarity among objects, is instrumental for the existence of stable matchings, among other uses.

In addition, we use a modification of the neutrality axiom that is restricted to sets in $\mathcal{X}_q$.

Fixed-Cardinality Neutrality: For all $A, B \in \mathcal{X}_q$, for all one-to-one mappings $\varphi: A \cup B \to X$,

$$\begin{aligned} &([xRy \Leftrightarrow \varphi(x)R\varphi(y) \text{ and } yRx \Leftrightarrow \varphi(y)R\varphi(x)] \text{ for all } x \in A, \text{ for all } y \in B) \\ &\quad \Rightarrow (A \succeq_q B \Leftrightarrow \varphi(A) \succeq_q \varphi(B) \text{ and } B \succeq_q A \Leftrightarrow \varphi(B) \succeq_q \varphi(A)). \end{aligned}$$

As shown in Bossert [25], these two axioms imply that comparisons of sets of cardinality q must be performed according to a rank-ordered lexicographic procedure. The class of these orderings represents a generalization of the leximin and the leximax orderings. Whereas leximin (leximax) proceeds by comparing worst (best) elements first and successively moves on to better and better (worse and worse) elements, the analogous comparisons according to the more general rank-ordered lexicographic rules can be performed in any order of rank. For instance, we could start by comparing the third-best elements in two sets and, in case of indifference, move on to the q-best elements, followed by the best elements, and so on.

To simplify the formal definition of these rules, suppose that, without loss of generality, the elements in $A = \{a_1, \ldots, a_q\} \in \mathcal{X}_q$ and $B = \{b_1, \ldots, b_q\} \in \mathcal{X}_q$ are numbered such that $a_i R a_{i+1}$ and $b_i R b_{i+1}$ for all $i \in \{1, \ldots, q-1\}$. Let Π^q denote the set of all permutations $\pi^q: \{1, \ldots, q\} \to \{1, \ldots, q\}$. The lexicographic rank-ordered rule $\succeq_q^{\pi^q}$ corresponding to $\pi^q \in \Pi^q$ is defined by letting, for all $A, B \in \mathcal{X}_q$,

$$\begin{aligned} A \succeq_q^{\pi^q} B \quad \Leftrightarrow \quad & (A = B \text{ or} \\ & [\exists i \in \{1, \ldots, q\} \text{ such that} \\ & a_{\pi^q(j)} = b_{\pi^q(j)} \text{ for all } j < i \text{ and } a_{\pi^q(i)} P b_{\pi^q(i)}]). \end{aligned}$$

A relation $\succeq_q$ on $\mathcal{X}_q$ is called a lexicographic rank-ordered rule if and only if there exists a $\pi^q \in \Pi^q$ such that $\succeq_q = \succeq_q^{\pi^q}$.

We now obtain the following characterization result (see Bossert [25]).

Theorem 24 *Suppose $q \in \mathbb{N}$, $|X| \geq 2q+1$, R is a linear ordering on X, and $\succeq_q$ is an ordering on $\mathcal{X}_q$. $\succeq_q$ satisfies responsiveness and fixed-cardinality neutrality if and only if $\succeq_q$ is a lexicographic rank-ordered rule.*

Proof. See Bossert [25]. ■

This result is not true if $q+1 < |X| < 2q+1$—see Bossert [25] for a discussion. In the trivial case where $|X| = q+1$, any two sets $A, B \in \mathcal{X}_q$ can differ in at most one element, and responsiveness immediately determines a linear ordering that coincides with $\succeq_q^{\pi^q}$ for all $\pi^q \in \Pi^q$.

Another set-ranking issue in the context of matching problems appears in Dutta and Massó [44]. They consider a problem of matching firms and workers where workers may care not only about the firm with which they are matched but also about who their co-workers are in a given match. The resulting preferences are assumed to have certain properties in Dutta and Massó [44]. The main message of that paper is a warning that stable matchings may fail to exist as soon as workers' preferences over firms exhibit a dependency on their sets of co-workers, even if the form of this dependence is quite specific and limited.

5.3 Additive Representability and Separability

As before, let X be the set of objects under consideration. For some applications where the elements of subsets of X are mutually compatible, it may be appropriate to include the empty set as a set to be ranked relative to other finite subsets of X. For example, if the sets to be ranked are additions to existing sets (such as potential new members of a club), the empty set can be interpreted as the decision to add nobody. In those cases, the objective is to establish a ranking $\succeq$ on $\mathcal{X}_0$ rather than just on $\mathcal{X}$.

A simple way to generate rankings of sets of mutually compatible objects from rankings of the objects themselves is the following. Choose a utility representation $U: X \to \mathbb{R}$ of an ordering R on X, provided that such a representation exists (this is guaranteed if the set X is finite or countably infinite). Additive representability means that the criterion used to rank any two sets according to $\succeq$ is the total utility of the elements of each set.

Additive Representability: There exists $U: X \to \mathbb{R}$ such that, for all $A, B \in \mathcal{X}$,

$$A \succeq B \Leftrightarrow \sum_{x \in A} U(x) \geq \sum_{x \in B} U(x). \tag{17.38}$$

If the empty set is included as a set to be ranked by $\succeq$, the above definition applies to all elements in $\mathcal{X}_0$, where the corresponding sum is replaced with zero if $A = \emptyset$ or $B = \emptyset$ in (17.38). The inclusion of the empty set provides, for example, a way to give meaning to the position of singletons relative to special situations. For example, the statement $\{x\} \succ \emptyset$ could be interpreted as saying that having $x \in X$ alone is better than nothing or, analogously, $\emptyset \succ \{x\}$ could mean that is better to have nothing than being stuck with x. A natural question would then be whether $\succeq$ can be generated by some utility representation of R in a way such that the sum of the utilities generated by the members of a set is used as a criterion to establish a set ranking.

We can view the characterization of those set rankings which admit an additive representation as part of a larger question. There is an important literature on the representability of binary relations over (finite) vectors. This covers, for example, the study of preferences over consumption vectors, where the value of each component of a vector stands for the consumption of the good identified with that component, and the dimension of the vector is the number of goods available in the economy. Many other interpretations are possible. In particular, any subset of a nonempty and finite set $X = \{x_1, \ldots, x_n\}$ can be identified with its n-dimensional characteristic vector, which is defined as follows. For each $i \in \{1, \ldots, n\}$, the characteristic vector of a set $A \in \mathcal{X}_0$ has a one in the i^{th} component if and only if $x_i \in A$ and a zero in the i^{th} component otherwise. Hence, a ranking of the elements of $\mathcal{X}_0$ can be obtained from a ranking of vectors of a given dimension whose components assume the values zero and one only.

The development of general results for additively representable preferences over vectors is beyond the scope of this survey but the special case relevant for set rankings is of interest. An excellent source for a detailed description of the general approach to the additive-representability problem can be found in Fishburn [49], in particular Chapter 4; see also, for example, Fishburn [48, 54], Fishburn and Roberts [55, 56], Krantz, Luce, Suppes, and Tversky [88], Roberts [124], Scott [132], and Scott and Suppes [133] for further discussions and results in that and related areas of research.

We have already remarked that, for the purposes of this survey, an important special case is obtained if the vectors to be ranked are characteristic vectors of sets. As explained in Fishburn [53], this is a basic problem in the theory of subjective probability if the requirement that the empty set is the worst possible set is added. Translated into our terminology, additive representability amounts to Savage's [130] definition of the existence of an agreeing measure, provided that the function $U : X \to \mathbb{R}$ in (17.38) has the properties of a probability measure. Under these assumptions, de Finetti [39] observes that (in addition to $\succeq$ being an ordering) the following conditions are necessary for additive representability by such a function U.

Nonnegativity: For all $A \in \mathcal{X}_0$,

$$A \succeq \emptyset.$$

Nontriviality: $X \succ \emptyset$.

Strong Extended Independence: For all $A, B \in \mathcal{X}_0$, for all $C \subseteq X \setminus (A \cup B)$,

$$A \succ B \Leftrightarrow A \cup C \succ B \cup C.$$

The latter axiom is a strengthening of extended independence as defined in Section 3. Savage [130] uses the terminology qualitative probability for an ordering $\succeq$ satisfying those three axioms. As stated above, these axioms are necessary. This raises the question whether they are also sufficient. This is not the case unless X contains four elements or less, as demonstrated in Kraft, Pratt, and Seidenberg [85]. Let $X = \{x_1, \ldots, x_5\}$ and consider an ordering $\succeq$ on $\mathcal{X}_0$ satisfying the above axioms such that $\{x_2, x_3, x_5\} \succ \{x_1, x_4\}$, $\{x_1, x_5\} \succ \{x_2, x_3\}$, $\{x_3, x_4\} \succ \{x_2, x_5\}$, and $\{x_2\} \succ \{x_3, x_5\}$. Additive representability requires the existence of a utility function U such that

$$\begin{aligned}
U(x_2) + U(x_3) + U(x_5) &> U(x_1) + U(x_4), \\
U(x_1) + U(x_5) &> U(x_2) + U(x_3), \\
U(x_3) + U(x_4) &> U(x_2) + U(x_5), \\
U(x_2) &> U(x_3) + U(x_5).
\end{aligned}$$

Adding those inequalities and simplifying immediately yields a contradiction.

To obtain sufficient conditions for additive representability, an infinite number of conditions can be invoked; see Fishburn [49] for details. Bounds on the number of such conditions to be tested are studied in Fishburn [53].

Additive representability is an assumption that is often made in different applications where rankings of sets are relevant. We mention a few. One application is within the theory of coalition formation. Consider, specifically, games where agents only care about the set of agents they form a coalition with. These have been called purely hedonic games. Banerjee, Konishi, and Sönmez [9] and Bogomolnaia and Jackson [23] have proven that these games can have no stable coalition structures, under different notions of stability, even if the preferences of each player over different coalitions are additively representable. In that application, the assumption of additive representability reinforces the negative character of the main conclusions, since they hold even under such a strong assumption. In other applications, like voting and matching, additive representabilty yields positive results. There, because it is considered a strong assumption, it is often relaxed to the related but weaker notion of separability, which we define below. The positive results quoted below, which are obtained under the assumption of separability, would also hold, a fortiori, should the relevant rankings of sets be additively representable.

Additive representability imposes strong requirements on preferences over sets. Separability is a natural weakening of this notion. For contexts where all finite subsets of X, including the empty set, can be properly interpreted and ranked, the information whether each singleton is better or worse than the empty set can be used. One can declare objects $x \in X$ such that $\{x\} \succ \emptyset$ to be 'desirable,' whereas those $x \in X$ with $\emptyset \succ \{x\}$ are thought of as 'undesirable.' The separability axiom then requires that 'desirability' or 'undesirability' of adding an object x to a set A is independent of the constituent elements of A.

Separability: For all $A \in \mathcal{X}$, for all $x \in X \setminus A$,

$$A \cup \{x\} \succ A \Leftrightarrow \{x\} \succ \emptyset.$$

Clearly, additive representability implies separability, but the converse is not true. For example, the ordering $\succeq$ on $\mathcal{X}_0$ with $X = \{x, y, z\}$ given by

$$\{x, y, z\} \succ \{x, y\} \succ \{x, z\} \succ \{y, z\} \succ \{z\} \succ \{y\} \succ \{x\} \succ \emptyset$$

is separable but not additively representable.

Separability conditions have proven useful in determining domain restrictions under which strategy-proof social choice functions can be defined. Barberà, Sonnenschein, and Zhou [15] consider a linear ordering that ranks sets of outcomes, but no ranking of the elements of X is included in their approach. They study the manipulability of voting schemes that select sets of outcomes and characterize a class of nonmanipulable voting schemes under the assumption that the voters' preferences that rank sets of alternatives are restricted to be separable according to the above definition; see their paper for details.

Although being weaker than additive representability, separability has strong implications as a condition of non-complementarity among the elements of a set. A similar notion is embodied in an analogous condition proposed by Benoît and Kornhauser [21] in order to analyze the relationships between rankings of candidates and rankings of assemblies. These authors treat a ranking of committees as a primitive and show that under a separability condition, this ranking of assemblies induces a ranking of possible assembly members in a natural way; see Benoît and Kornhauser [21] for details.

Some of the matching literature discusses other restrictions on the preferences of agents over sets, and some of these conditions are formulated in choice-theoretic terms rather than directly in terms of set rankings. A well-known example is a substitutability axiom employed in Roth and Sotomayor [128], and other requirements in the same spirit can be found in Alkan [1] and in Martínez, Massó, Neme, and Oviedo [94]. Their requirements on choice functions would be satisfied by additively representable preferences. Although the choice-theoretic requirements are weaker, they once again prevent complementarities among the members of a set.

5.4 Signed Orderings

Fishburn [52] analyzes the problem of deriving preferences on sets of compatible outcomes (such as committees) in an informationally enriched framework. Rather than assuming that the only information available when establishing the extension is an ordering on the set of alternatives themselves, Fishburn [52] considers situations where a signed ordering is available. Given the set X of alternatives, define the set X^* by letting $x^* \in X^*$ if and only if $x \in X$. The elements of X^* can be interpreted as the 'complements' of the alternatives in X, and it is assumed that $(x^*)^* = x$ for all $x \in X$. A signed ordering is an ordering defined on the set $X \cup X^*$, that is, a reflxive, transitive, and complete binary relation $R^* \subseteq (X \cup X^*) \times (X \cup X^*)$. The richer informational contents of the relation R^* permits us to go beyond mere comparisons of the alternatives according to their relative desirability. In particular, the interpretation of R^* is as follows (see Fishburn [52]). For any two $x, y \in X$, xP^*y means that it is more desirable to have x as a member of the committee than having y on the committee; x^*P^*y means that it is more important to prevent x from being on the comittee than having y on the committee; xP^*y^* means that having x on the committee is preferred to leaving y off the committee; and $x^*P^*y^*$ means that it is more important to leave x off the committee than preventing y from being on the committee.

It is assumed that the ordering R^* is self-reflecting, which requires the following.

Self-Reflection: For all $a, b \in X \cup X^*$,

$$aR^*b \Leftrightarrow b^*R^*a^*.$$

A self-reflecting signed ordering allows us to partition the set X into the set of 'desirable' elements X^d, the set of 'undesirable' elements X^u, and the set of 'indifferent' elements X^i by letting $X^d = \{x \in X \mid xP^*x^*\}$, $X^u = \{x \in X \mid x^*P^*x\}$, and $X^i = \{x \in X \mid xI^*x^*\}$. Thus, signed orderings provide an alternative to the use of the empty set to express the notion of desirable and undesirable objects.

Now consider a relation $\succeq$ on $\mathcal{X}$. Based on the signed ordering R^*, desirable, undesirable, and indifferent committees can be identified. One suggestion of doing so, due to Fishburn [52], proceeds as follows. To begin with, all nonempty subsets of X^d can clearly be considered desirable. The same is true for subsets of $X^d \cup X^i$ that have a nonempty intersection with X^d. Fishburn [52] suggests to include, in addition, sets such as $\{x, y\}$ if $x \in X^d$, $y \in X^u$, and xP^*y^* because, according to the interpretation of the signed ordering R^*, including x is more important than excluding y and, therefore, the committee $\{x, y\}$ is worth having. By generalizing this argument to sets with an arbitrary number of elements, we obtain the following definition of the set $\mathcal{X}^d$ of desirable committees. For all $A \in \mathcal{X}$,

$$\begin{aligned} A \in \mathcal{X}^d \quad \Leftrightarrow \quad & \Big(A \cap X^d \neq \emptyset \text{ and } \Big[A \cap X^u = \emptyset \text{ or} \\ & \big(\text{there exists a one-to-one mapping} \\ & \varphi\colon A \cap X^u \to A \cap X^d \text{ such that} \\ & [\varphi(x)R^*x^* \text{ for all } x \in A \cap X^u \text{ and} \\ & ([\text{there exists } x \in A \cap X^u \text{ such that } \varphi(x)P^*x^*] \\ & \text{ or } |A \cap X^d| > |A \cap X^u|)])\Big]\Big). \end{aligned}$$

Analogously, we can define the set $\mathcal{X}^u$ of undesirable committees and the set $\mathcal{X}^i$ of indifferent committees by letting, for all $A \in \mathcal{X}$,

$$\begin{aligned} A \in \mathcal{X}^u \quad \Leftrightarrow \quad & \Big(A \cap X^u \neq \emptyset \text{ and } \Big[A \cap X^d = \emptyset \text{ or} \\ & \big(\text{there exists a one-to-one mapping} \\ & \varphi\colon A \cap X^d \to A \cap X^u \text{ such that} \\ & [\varphi(x)^*R^*x \text{ for all } x \in A \cap X^d \text{ and} \\ & ([\text{there exists } x \in A \cap X^d \text{ such that } \varphi(x)^*P^*x] \\ & \text{ or } |A \cap X^u| > |A \cap X^d|)])\Big]\Big) \end{aligned}$$

and

$$\begin{aligned} A \in \mathcal{X}^i \quad \Leftrightarrow \quad & \Big(|A \cap X^d| = |A \cap X^u| \text{ and } \Big[A \cap X^d = \emptyset \text{ or} \\ & \big(\text{there exists a bijective mapping} \\ & \varphi\colon A \cap X^u \to A \cap X^d \text{ such that} \\ & [\varphi(x)I^*x^* \text{ for all } x \in A \cap X^u)\Big]\Big). \end{aligned}$$

Clearly, $\{\mathcal{X}^d, \mathcal{X}^u, \mathcal{X}^i\}$ is, in general, not a partition of $\mathcal{X}$—the three sets are disjoint but their union may be a strict subset of $\mathcal{X}$. See Fishburn [52] for examples. However, the sets can nevertheless be used to provide a first step towards establishing a ranking $\succeq$ on $\mathcal{X}$. In particular, it is natural to require that a relation $\succeq$ on $\mathcal{X}$ satisfies the requirement that if a set $A \in \mathcal{X}$ is desirable and $B \in \mathcal{X}$ is indifferent or undesirable or A is indifferent and B is undesirable, then A should be better than B according to $\succeq$. Analogously, if both A and B are in $\mathcal{X}^i$, it is reasonable to postulate that they should be declared indifferent according to $\succeq$. Formally, for all $A, B \in \mathcal{X}$,

$$\big[(A \in \mathcal{X}^d \text{ and } B \in \mathcal{X}^i \cup \mathcal{X}^u) \text{ or } (A \in \mathcal{X}^i \text{ and } B \in \mathcal{X}^u)\big] \Rightarrow A \succ B \tag{17.39}$$

and

$$A, B \in \mathcal{X}^i \Rightarrow A \sim B. \tag{17.40}$$

If $A \sim A$ for all $A \in \mathcal{X}$ is added to the relations determined by (17.39) and (17.40), the resulting relation $\succeq$ is a quasi-ordering on $\mathcal{X}$ that is based on the properties of the underlying signed ordering R^* in a very intuitive way. See Fishburn [52] for possible ways of extending this quasi-ordering to an ordering and for a more detailed discussion of the use of signed orderings.

6 CONCLUDING REMARKS

The initial purpose of this chapter was to bring together a number of lines of research which are scattered around in the literature on economics, decision theory, and moral philosophy. Our emphasis has been on the fact that these inquiries share a common tool, the ranking of sets, as the way to express a wide variety of concerns. We have emphasized the importance of combining the axiomatic analysis with appropriate interpretations in order to establish the connections, as well as the boundaries, between these rather disparate sources. Rankings play a more important role than their numerical representations in the majority of the works we have discussed. Hence, the inclusion of this survey in a volume on utility theory is probably better viewed as an attempt to gather together a number of interesting treatments of preferences which might otherwise have been fogotten because they are somewhat nonstandard. We hope that those who are already familiar with at least some aspects of the issues discussed here will find this chapter a useful survey of the state of the art.

In addition, we may also have attracted the interest of readers that are new to this field of research and have provided them with a summary of some of the major recent advances.

Finally, we hope to have generated interest from those readers who are engaged in fields of research that can benefit from the literature on ranking sets. We hope that, for example, researchers in the areas discussed in Section 2 (the theory of voting, matching problems, and the intersection of economics and philosophy) will find some of the results presented here to be useful.

Acknowledgements

Financial support through an Acciones Integradas grant from the British Council and the Spanish Ministry of Culture and Education is gratefully acknowledged.

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